

# ON THE COMPLETENESS AND RIESZ BASIS PROPERTY OF ROOT SUBSPACES OF BOUNDARY VALUE PROBLEMS FOR FIRST ORDER SYSTEMS AND APPLICATIONS

ANTON A. LUNYOV AND MARK M. MALAMUD

**ABSTRACT.** The paper is concerned with the completeness property of root functions of general boundary value problems for  $n \times n$  first order systems of ordinary differential equations on a finite interval. In comparison with the recent paper [45] we substantially relax the assumptions on boundary conditions guarantying the completeness of root vectors, allowing them to be non-weakly regular and even degenerate. Emphasize that in this case the completeness property substantially depends on the values of a potential matrix at the endpoints of the interval.

It is also shown that the system of root vectors of the general  $n \times n$  Dirac type system subject to certain boundary conditions forms a Riesz basis with parentheses. We also show that arbitrary complete dissipative boundary value problem for Dirac type operator with a summable potential matrix admits the spectral synthesis in  $L^2([0, 1]; \mathbb{C}^n)$ . Finally, we apply our results to investigate completeness and the Riesz basis property of the dynamic generator of spatially non-homogenous damped Timoshenko beam model.

## CONTENTS

1. Introduction	2
2. Preliminaries	5
3. Asymptotic behavior of solutions and characteristic determinant	12
4. Explicit completeness results	20
4.1. Explicit sufficient conditions of completeness.	20
4.2. Example	23
4.3. Necessary conditions of completeness.	24
5. The Riesz basis property for root functions	26
6. General properties of the resolvent and spectral synthesis	32
6.1. General properties of the resolvent	32
6.2. Spectral synthesis for dissipative Dirac type operators	38
7. Application to the Timoshenko beam model	41
References	48

---

2010 *Mathematics Subject Classification.* 47E05, 34L10, 35L35, 47A15.

*Key words and phrases.* Systems of ordinary differential equations; regular boundary conditions; completeness of root vectors; Riesz basis property; resolvent operator; dissipative operators; spectral synthesis; Timoshenko beam model.

## 1. INTRODUCTION

Spectral theory of non-selfadjoint boundary value problems (BVP) on a finite interval  $\mathcal{I} = (a, b)$  for  $n$ th order ordinary differential equations (ODE)

$$y^{(n)} + q_1 y^{(n-2)} + \dots + q_{n-1} y = \lambda^n y, \quad x \in (a, b), \quad (1.1)$$

with coefficients  $q_j \in L^1(a, b)$  takes its origin in the classical papers by Birkhoff [8, 9] and Tamarkin [64, 65, 66]. They introduced the concept of *regular boundary conditions* for ODE and investigated the asymptotic behavior of eigenvalues and eigenfunctions of related BVP. Moreover, they proved that the system of root functions, i.e. eigenfunctions and associated functions, of the regular BVP is complete. Their results are also treated in the classical monographs (see [52, Section 2] and [22, Chapter 19]).

The completeness property of non-regular BVP for  $n$ th order ODE (1.1) has been studied by M.V. Keldysh [30], A.A. Shkalikov [56], A.G. Kostyuchenko and A.A. Shkalikov [35], G.M. Gubreev [28], A.P. Khromov [32, 33], V.S. Rykhlov [55] and many others (see references in [33]). On the other hand, the Riesz basis property for regular BVP were investigated by N. Dunford [20], V.P. Mikhailov [50], G.M. Kesel'man [31], N. Dunford and J. Schwartz [22, Chapter 19.4], A.A. Shkalikov [57, 58, 59]. Numerous papers are devoted to the completeness and Riesz basis property for the Sturm-Liouville operator (see the recent paper [61] by A. Shkalikov and O. Veliev and the review [40] by A.S. Makin and the references therein). We especially mention the recent achievements for periodic (anti-periodic) Sturm-Liouville operator  $-\frac{d^2}{dx^2} + q(x)$  on  $[0, \pi]$ . Namely, F. Gesztesy and V.A. Tkachenko [24, 25] for  $q \in L^2[0, \pi]$  and later on P. Djakov and B.S. Mityagin [18] for  $q \in W^{-1,2}[0, \pi]$  established by different methods a *criterion* for the system of root functions to contain a Riesz basis (see Remark 5.11 for detailed discussion).

In this paper we consider first order system of ODE of the form

$$Ly := L(Q)y := -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, \dots, y_n), \quad (1.2)$$

where  $B$  is a nonsingular diagonal  $n \times n$  matrix with complex entries,

$$B = \text{diag}(b_1, b_2, \dots, b_n) \in \mathbb{C}^{n \times n}, \quad (1.3)$$

and  $Q(\cdot) =: (q_{jk}(\cdot))_{j,k=1}^n \in L^1([0, 1]; \mathbb{C}^{n \times n})$  is a potential matrix.

Note that, systems (1.2) form a more general object than ordinary differential equations. Namely, the  $n$ th order ODE (1.1) can be reduced to the system (1.2) with  $b_j = \exp(2\pi i j/n)$  (see [41]). Nevertheless, in general a BVP for ODE (1.1) is not reduced to a BVP (1.2)–(1.4) (see below). Systems (1.2) are of significant interest in some theoretical and practical questions.

For instance, if  $n = 2m$ ,  $B = \text{diag}(-I_m, I_m)$  and  $Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}$ , system (1.2) is equivalent to the Dirac system (see [37, Section VII.1], [46, Section 1.2]). Note also that equation (1.2) is used to integrate the  $N$ -waves problem arising in nonlinear optics [54, Sec. III.4].

With system (1.2) one associates, in a natural way, the maximal operator  $L = L(Q)$  acting in  $L^2([0, 1]; \mathbb{C}^n)$  on the domain

$$\text{dom}(L) = \{y \in W^{1,1}([0, 1]; \mathbb{C}^n) : Ly \in L^2([0, 1]; \mathbb{C}^n)\}.$$

To obtain a BVP, equation (1.2) is subject to the following boundary conditions

$$Cy(0) + Dy(1) = 0, \quad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n \times n}. \quad (1.4)$$

Denote by  $L_{C,D} := L_{C,D}(Q)$  the operator associated in  $L^2([0, 1]; \mathbb{C}^n)$  with the BVP (1.2)–(1.4). It is defined as the restriction of  $L = L(Q)$  to the domain

$$\text{dom}(L_{C,D}) = \{y \in \text{dom}(L) : Cy(0) + Dy(1) = 0\}. \quad (1.5)$$

Moreover, in what follows we always impose the maximality condition

$$\text{rank} \begin{pmatrix} C & D \end{pmatrix} = n, \quad (1.6)$$

which is equivalent to  $\ker(CC^* + DD^*) = \{0\}$ .

To the best of our knowledge, the spectral problem (1.2)–(1.4) has first been investigated by G.D. Birkhoff and R.E. Langer [10]. Namely, they have extended some previous results of Birkhoff and Tamarkin on non-selfadjoint boundary value problem for ODE (1.1) to the case of BVP (1.2)–(1.4). More precisely, they introduced the concepts of *regular and strictly regular boundary conditions* (1.4) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding operator  $L_{C,D}$ . Moreover, they proved a *pointwise convergence result* on spectral decompositions of the operator  $L_{C,D}$  corresponding to the BVP (1.2)–(1.4) with regular boundary conditions.

The problem of the completeness of the system of root functions of *general BVP* (1.2)–(1.4) has first been investigated in the recent papers [43, 44, 45] by one of the authors and L.L. Oridoroga. In these papers the concept of *weakly regular* boundary conditions for the system (1.2) was introduced and the completeness of root vectors for this class of BVP was proved. During the last decade there appeared numerous papers devoted mainly to the Riesz basis property for  $2 \times 2$  Dirac system subject to the *regular or strictly regular* boundary conditions (see [69, 70, 51, 12, 29, 7, 13, 14, 16, 17, 18, 19]).

Let us recall the definition of regular (see [10, p. 89]) and weakly regular (see [43, 45]) boundary conditions. To this end we need the following construction. Let  $A = \text{diag}(a_1, \dots, a_n)$  be a diagonal matrix with entries  $a_k$  (not necessarily distinct) that are not lying on the imaginary axis,  $\text{Re } a_k \neq 0$ . Starting from arbitrary matrices  $C, D \in \mathbb{C}^{n \times n}$ , we define the auxiliary  $n \times n$  matrix  $T_A(C, D)$  as follows:

- if  $\text{Re } a_k < 0$ , then the  $k$ th column in the matrix  $T_A(C, D)$  coincides with the  $k$ th column of the matrix  $C$ ,
- if  $\text{Re } a_k > 0$ , then the  $k$ th column in the matrix  $T_A(C, D)$  coincides with the  $k$ th column of the matrix  $D$ .

Now consider the lines

$$l_j := \{\lambda \in \mathbb{C} : \text{Re}(ib_j \lambda) = 0\}, \quad j \in \{1, \dots, n\}, \quad (1.7)$$

of the complex plane. They divide the complex plane into  $m = 2r \leq 2n$  sectors. Denote these sectors by  $\sigma_1, \sigma_2, \dots, \sigma_m$ . Let  $z_j$  lie in the interior of  $\sigma_j, j \in \{1, \dots, m\}$ . The boundary conditions (1.4) are called regular whenever

$$\det T_{iz_j B}(C, D) \neq 0, \quad j \in \{1, \dots, m\}. \quad (1.8)$$

We call  $z \in \mathbb{C}$  *admissible* if  $\text{Re}(ib_j z) \neq 0$  for  $j \in \{1, \dots, n\}$ . Since  $T_{iz_j B}(C, D)$  does not depend on a particular choice of the point  $z_j \in \sigma_j$ , the boundary conditions (1.4) are regular if and only if  $\det T_{iz B}(C, D) \neq 0$  for each admissible  $z$ .

**Definition 1.1.** ([45]) *The boundary conditions (1.4) are called weakly B-regular (or, simply, weakly regular) if there exist three admissible complex numbers  $z_1, z_2, z_3$  satisfying the following conditions:*

- (a) *the origin is an interior point of the triangle  $\Delta_{z_1 z_2 z_3}$ ;*

(b)  $\det T_{iz_j B}(C, D) \neq 0$  for  $j \in \{1, 2, 3\}$ .

In the case of Dirac type system ( $B = B^*$ ) the weak regularity of boundary conditions (1.4) is equivalent to their regularity (1.8) and turns into

$$\det T_{\pm} := \det(CP_{\mp} + DP_{\pm}) \neq 0. \quad (1.9)$$

Here  $P_+$  and  $P_-$  denote the spectral projections onto "positive" and "negative" parts of the spectrum of  $B = B^*$ , respectively. Therefore, by [45, Theorem 1.2], this condition implies the completeness and minimality in  $L^2([0, 1]; \mathbb{C}^n)$  of the root functions of BVP (1.2)–(1.4). In special cases this statement has earlier been obtained by V.A. Marchenko [46, §1.3] ( $2 \times 2$  Dirac system) and V.P. Ginzburg [26] ( $B = I_n, Q = 0$ ).

Our first main result (Theorem 4.1) states the completeness property for the general BVP (1.2)–(1.4) with non-weakly regular boundary conditions. It substantially generalizes the corresponding results from [45] and [1]. *Emphasize that in the case of non-weakly regular boundary conditions the completeness property substantially depends on the values  $Q(0)$  and  $Q(1)$ .* The latter means that Theorem 4.1 cannot be treated as a perturbation theory result: the operator  $L_{C,D}(Q)$  satisfying the conditions of this theorem is complete while the system of root vectors of the unperturbed operator  $L_{C,D}(0)$  may have infinite defect in  $L^2([0, 1]; \mathbb{C}^{n \times n})$ . We demonstrate this fact by the corresponding examples (cf. Corollary 4.7).

Our second main achievement is the Riesz basis property for general  $n \times n$  Dirac type system with  $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$  subject to certain boundary conditions. These conditions form rather broad class that covers, in particular, periodic, antiperiodic, and regular splitting (not necessarily selfadjoint) boundary conditions for  $2n \times 2n$  Dirac system ( $B = \text{diag}(-I_n, I_n)$ ) (see Theorem 5.6 and Proposition 5.8 for the precise statements). *Emphasize that to the best of our knowledge even for  $2n \times 2n$  Dirac systems with  $n > 1$  the results on the Riesz basis property are obtained here for the first time.*

In this connection we mention the series of recent papers by P. Djakov and B.S. Mityagin [14, 16, 17, 18, 19]. In [14] the authors proved that the system of root functions for  $2 \times 2$  Dirac system with  $Q \in L^2([0, 1]; \mathbb{C}^{2 \times 2})$  subject to the *regular* boundary conditions forms a *Riesz basis with parentheses* while this system forms *ordinary Riesz basis* provided that the boundary conditions are *strictly regular*. Moreover, in [16, Theorem 13], [18, Theorem 19] and [19] it is established a *criterion* for the system of root functions to contain a Riesz basis for periodic (resp., antiperiodic)  $2 \times 2$  Dirac operator in terms of the Fourier coefficients of  $Q$  as well as in terms of periodic (resp., antiperiodic) and Dirichlet spectra.

Further, it is worth to mention one more our result: any dissipative BVP (1.2)–(1.4) admits the spectral synthesis in  $L^2([0, 1]; \mathbb{C}^n)$  (see Theorem 6.12). The spectral synthesis problem was originated by J. Wermer [71] and then studied by many authors (see [11, 47, 53] and references therein). We also mention recent preprints [3, 4, 5, 6] devoted to the problems of completeness and spectral synthesis for singular perturbations of selfadjoint operators and systems of exponents and to problems of removal of spectrum.

Note in this connection that each dissipative boundary value problem for equation (1.1) with  $n \geq 2$  admits the spectral synthesis. The proof of this fact substantially relies on two main ingredients:

- the resolvent of any BVP for equation (1.1) is the trace class operator;
- the dissipative boundary value problem is always complete

(see Remark 6.17 for details).

As distinct from the situation above, the resolvent of the Dirac type operator  $L_{C,D}(Q)$  is no longer in trace class (see Proposition 6.7). Moreover, the system of root vectors of the dissipative operator  $L_{C,D}(Q)$  may be incomplete (see, for instance [45, Remark 5.10]). Thus, the problem of spectral synthesis is non-trivial in this case.

Finally, we apply our main abstract results with  $B = B^* \in \mathbb{C}^{4 \times 4}$  to the Timoshenko beam model investigated under the different restrictions in numerous papers (see [67, 68, 34, 62, 63, 74, 73, 72] and the references therein). We show in Proposition 7.1 that the dynamic generator of this model is similar to the special  $4 \times 4$  Dirac type operator. It allows us to derive completeness property in both regular and non-regular cases. Moreover, *in the regular case we obtain also the Riesz basis property with parentheses*.

The paper is organized as follows. In Section 2 we obtain the general result on completeness that generalizes [45, Theorem 1.2]. In Section 3 we obtain refined asymptotic formulas for solutions of system (1.2) and the characteristic determinant  $\Delta(\cdot)$  of the problem (1.2)–(1.4), provided that the potential matrix  $Q(\cdot)$  is continuous at the endpoints 0 and 1.

In Section 4 we prove our main result on completeness, Theorem 4.1. We illustrate this result in  $2 \times 2$  case by deriving completeness and minimality in  $L^2([0, \pi]; \mathbb{C}^2)$  of the system

$$\left\{ \text{col}(e^{anx} \sin nx, ne^{(a-i)nx}) \right\}_{n \in \mathbb{Z} \setminus \{0\}}. \quad (1.10)$$

We also obtain some necessary conditions on completeness for general BVP (1.2)–(1.4) generalizing [45, Proposition 5.12] and coinciding with it in the case of  $2 \times 2$  Dirac system.

In Section 5 we prove the mentioned above results on the Riesz basis property with parentheses for BVP (1.2)–(1.4) with a bounded potential matrix. In Section 6 we discuss different properties of the resolvent operator  $(L_{C,D}(Q) - \lambda)^{-1}$ . In particular, we show that the resolvent difference of two operators  $L_{C_1,D_1}(Q_1)$  and  $L_{C_2,D_2}(Q_2)$  is trace class operator (Theorem 6.3). Using this result we prove mentioned above result on spectral synthesis for dissipative Dirac type operators as well as obtain some explicit conditions in terms of the matrices  $B, C, D, Q(\cdot)$  for the operator  $L_{C,D}(Q)$  to admit the spectral synthesis (see Theorem 6.16).

Finally, in Section 7 we prove mentioned above results on the completeness and Riesz basis property with parentheses for the dynamic generator of spatially non-homogenous Timoshenko beam model with both boundary and locally distributed damping.

The main results of Sections 2–4 have been announced in [38].

**Notation.**  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{C}^n$ ;  $\mathbb{C}^{n \times n}$  denotes the set of  $n \times n$  matrices with complex entries.  $I_n (\in \mathbb{C}^{n \times n})$  denotes the identity matrix;  $\text{GL}(n, \mathbb{C})$  denotes the set of nonsingular matrices from  $\mathbb{C}^{n \times n}$ ;  $W^{n,p}[a, b]$  is Sobolev space of functions  $f$  having  $n-1$  absolutely continuous derivatives on  $[a, b]$  and satisfying  $f^{(n)} \in L^p[a, b]$ .

$T$  is a closed operator in a Hilbert space  $\mathfrak{H}$ ;  $\sigma(T)$  and  $\rho(T) = \mathbb{C} \setminus \sigma(T)$  denote the spectrum and resolvent set of the operator  $T$ , respectively.

$\mathfrak{S}_p(\mathfrak{H})$ ,  $0 < p \leq \infty$ , denotes the Neumann-Schatten ideals in a Hilbert space  $\mathfrak{H}$ . In particular,  $\mathfrak{S}_\infty(\mathfrak{H})$  is the ideal of compact operators.  $\mathfrak{S}_p(\mathfrak{H})$  is a two-sided ideal in algebra  $\mathcal{B}(\mathfrak{H})$  of bounded linear operators.

## 2. PRELIMINARIES

In what follows we will systematically use the following simple lemma.

**Lemma 2.1.** *Let  $L_{C,D}(Q)$  be the operator defined by (1.2)–(1.6). Then there exist matrices  $C_*, D_* \in \mathbb{C}^{n \times n}$  such that  $\text{rank}(C_* \ D_*) = n$  and the adjoint operator  $L_{C,D}^* := (L_{C,D}(Q))^*$*

coincides with the restriction of the maximal differential operator

$$\begin{aligned} L_* y &:= -i(B^*)^{-1} y' + Q^*(x)y, \\ \text{dom}(L_*) &= \{y \in AC([0, 1]; \mathbb{C}^n) : L_* y \in L^2([0, 1]; \mathbb{C}^n)\}, \end{aligned} \quad (2.1)$$

to the domain

$$\text{dom}(L_{C,D}^*) = \{y \in \text{dom}(L_*) : C_* y(0) + D_* y(1) = 0\}. \quad (2.2)$$

In particular, if  $B = B^*$  then  $L_{C,D}^* = L_{C_*, D_*}(Q^*)$ .

Let  $\beta_1, \dots, \beta_r$  be all different values among  $b_1, \dots, b_n$ . Note that the lines

$$l_{jk} := \{\lambda \in \mathbb{C} : \text{Re}(i\beta_j \lambda) = \text{Re}(i\beta_k \lambda)\}, \quad 1 \leq j < k \leq r, \quad (2.3)$$

together with the lines

$$l_j := \{\lambda \in \mathbb{C} : \text{Re}(i\beta_j \lambda) = 0\}, \quad j \in \{1, \dots, r\}, \quad (2.4)$$

separate  $\nu \leq r^2 + r$  open sectors  $S_p$  with vertexes at the origin, such that for any  $p \in \{1, \dots, \nu\}$  the numbers  $\beta_1, \dots, \beta_r$  can be renumbered so that the following inequalities hold:

$$\text{Re}(i\beta_{j_1} \lambda) < \dots < \text{Re}(i\beta_{j_\kappa} \lambda) < 0 < \text{Re}(i\beta_{j_{\kappa+1}} \lambda) < \dots < \text{Re}(i\beta_{j_r} \lambda), \quad \lambda \in S_p. \quad (2.5)$$

Here  $\kappa = \kappa_p$  is the number of negative values among  $\text{Re}(i\beta_1 \lambda), \dots, \text{Re}(i\beta_r \lambda)$  in the sector  $S_p$ . We call  $z \in \mathbb{C}$  *feasible* if  $z$  does not belong to any of the lines (2.3) and (2.4), that is,  $z$  lies strictly inside some sector  $S_p$ . Note that feasible point is more restrictive notion than admissible point.

Clearly, each of the sectors  $S_p$  is of the form  $S_p = \{z : \varphi_{1p} < \arg z < \varphi_{2p}\}$ . Denote by  $S_{p,\varepsilon}$  a sector strictly embedded into the latter, i.e.,

$$S_{p,\varepsilon} := \{z : \varphi_{1p} + \varepsilon < \arg z < \varphi_{2p} - \varepsilon\}, \quad \text{where } \varepsilon > 0 \text{ is sufficiently small}; \quad (2.6)$$

$$S_{p,\varepsilon,R} := \{z \in S_{p,\varepsilon} : |z| > R\}. \quad (2.7)$$

**Proposition 2.2.** [45, Proposition 2.2] *Let  $\delta_{jk}$  be a Kronecker symbol, let*

$$B = \text{diag}(\beta_1 I_{n_1}, \dots, \beta_r I_{n_r}), \quad n_1 + \dots + n_r = n, \quad (2.8)$$

$$Q = (Q_{jk})_{j,k=1}^r, \quad Q_{jk} \in L^1([0, 1]; \mathbb{C}^{n_j \times n_k}), \quad (2.9)$$

$$Q_{jj}(\cdot) \equiv 0, \quad j \in \{1, \dots, r\}. \quad (2.10)$$

*Let also  $p \in \{1, \dots, \nu\}$  and let  $\varepsilon > 0$  be sufficiently small. Then for a sufficiently large  $R$ , equation (1.2) has a fundamental matrix solution*

$$Y(x, \lambda) = (Y_1 \ \dots \ Y_n), \quad Y_k(x, \lambda) = \text{col}(y_{1k}, \dots, y_{nk}), \quad k \in \{1, \dots, n\}, \quad (2.11)$$

*which is analytic in  $\lambda \in S_{p,\varepsilon,R}$  and satisfies (uniformly in  $x \in [0, 1]$ )*

$$y_{jk}(x, \lambda) = (\delta_{jk} + o(1))e^{ib_k \lambda x}, \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon,R}, \quad j, k \in \{1, \dots, n\}. \quad (2.12)$$

In what follows we will systematically use a concept of the similarity of unbounded operators.

**Definition 2.3.** *Let  $\mathfrak{H}_j$  be a Hilbert space,  $A_j$  a closed operator in  $\mathfrak{H}_j$  with domain  $\text{dom}(A_j)$ ,  $j \in \{1, 2\}$ . The operators  $A_1$  and  $A_2$  are called similar if there exists a bounded operator  $T$  (a similarity transformation operator) from  $\mathfrak{H}_1$  onto  $\mathfrak{H}_2$  with bounded inverse, such that  $A_2 = TA_1 T^{-1}$ , i.e.*

$$\text{dom}(A_2) = T \text{dom}(A_1) \quad \text{and} \quad A_2 f = T A_1 T^{-1} f, \quad f \in \text{dom}(A_2). \quad (2.13)$$



Note that similar operators  $A_1$  and  $A_2$  ( $A_2 = TA_1T^{-1}$ ) have the same spectra, algebraic and geometric multiplicities of eigenvalues, while the systems of their root vectors  $\{e_k^{(j)}\}$ ,  $j \in \{1, 2\}$ , are related by  $e_k^{(2)} = Te_k^{(1)}$ . Therefore, they also have the same geometric properties (completeness, minimality, basis property, etc.).

Let  $\Phi(x, \lambda)$  be a fundamental matrix solution of equation (1.2) satisfying

$$\Phi(0, \lambda) = I_n, \quad \lambda \in \mathbb{C}. \quad (2.14)$$

The characteristic determinant  $\Delta(\cdot)$  of the problem (1.2)–(1.4) is given by

$$\Delta(\lambda) := \det(C + D\Phi(1, \lambda)), \quad \lambda \in \mathbb{C}. \quad (2.15)$$

Next we prove the completeness result which slightly generalizes [45, Theorem 1.2].

**Theorem 2.4.** *Let  $Q(\cdot) \in L^1([0, 1]; \mathbb{C}^{n \times n})$ . Assume that there exist  $C, R > 0$ ,  $s \in \mathbb{Z}_+$  and three feasible numbers  $z_1, z_2, z_3$  satisfying the following conditions:*

- (i) *the origin is the interior point of the triangle  $\Delta_{z_1 z_2 z_3}$ ;*
- (ii) *for  $k \in \{1, 2, 3\}$  we have*

$$|\Delta(\lambda)| \geq \frac{Ce^{\operatorname{Re}(i\tau_k \lambda)}}{|\lambda|^s}, \quad \tau_k = \sum_{\substack{j=1 \\ \operatorname{Re}(ib_j z_k) > 0}}^n b_j, \quad |\lambda| > R, \quad \arg \lambda = \arg z_k. \quad (2.16)$$

*Then the system of root functions of the BVP (1.2)–(1.4) (of the operator  $L_{C,D}(Q)$ ) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .*

**Remark 2.5.** *In the case  $s = 0$  Theorem 2.4 is implicitly contained in [45, Theorem 1.2]. Our proof follows the scheme proposed in [45, Theorem 1.2].*

*Proof of Theorem 2.4.* By renumbering  $y_1, \dots, y_n$  we can assume that the matrix  $B$  satisfies (2.8) and hence  $Q$  has representation (2.9). Let

$$Q_1(x) := \operatorname{diag}(Q_{11}(x), \dots, Q_{rr}(x)) \quad (2.17)$$

and let  $W(\cdot)$  be the solution to the Cauchy problem

$$iB^{-1}W' = Q_1(x)W, \quad W(0) = I_n. \quad (2.18)$$

Due to the block structure of the matrices  $B$  and  $Q_1$  one easily derives

$$W(x) = \operatorname{diag}(W_{11}(x), \dots, W_{rr}(x)), \quad W_{jj}(x) \in \operatorname{GL}(n_j, \mathbb{C}), \quad x \in [0, 1]. \quad (2.19)$$

Denoting by  $W : y \rightarrow W(x)y$  the gauge transform and letting

$$\tilde{D} := DW(1) \quad \text{and} \quad \tilde{Q}(x) = W^{-1}(x)(Q(x) - Q_1(x))W(x) =: (\tilde{q}_{jk}(x))_{j,k=1}^n \quad (2.20)$$

we get

$$L_{C,\tilde{D}}(\tilde{Q}) = W^{-1}L_{C,D}(Q)W, \quad (2.21)$$

i.e.  $L_{C,D}(Q)$  and  $L_{C,\tilde{D}}(\tilde{Q})$  are similar. Clearly,  $\tilde{\Phi} =: W^{-1}\Phi$  is a fundamental solution of equation (1.2) with  $\tilde{Q}$  in place of  $Q$  and the corresponding characteristic determinant  $\tilde{\Delta}(\cdot)$  (see (2.15)) is

$$\tilde{\Delta}(\lambda) := \det(C + \tilde{D}\tilde{\Phi}(1, \lambda)) = \det(C + DW(1)W^{-1}(1)\Phi(1, \lambda)) = \Delta(\lambda). \quad (2.22)$$

So, above gauge transformation does not change the characteristic determinant. Therefore, replacing if necessary,  $L_{C,D}(Q)$  by  $L_{C,\tilde{D}}(\tilde{Q})$  we can assume that conditions (2.8)–(2.10) are satisfied.

Further, let  $\Psi(x, \lambda)$  be a fundamental  $n \times n$  matrix solution of equation (1.2) in a domain  $S$ , i.e.

$$\det(\Psi(0, \lambda)) \neq 0, \quad \lambda \in S. \quad (2.23)$$

Denote by  $\Psi_k(x, \lambda)$  the  $k$ th vector column of the matrix  $\Psi(x, \lambda)$ , i.e.,

$$\Psi(x, \lambda) = (\Psi_1 \quad \dots \quad \Psi_n), \quad \Psi_k(x, \lambda) = \text{col}(\psi_{1k}, \dots, \psi_{nk}). \quad (2.24)$$

Further, denote

$$A_\Psi(\lambda) := C\Psi(0, \lambda) + D\Psi(1, \lambda), \quad (2.25)$$

$$\Delta_\Psi(\lambda) := \det A_\Psi(\lambda) = \det(C\Psi(0, \lambda) + D\Psi(1, \lambda)). \quad (2.26)$$

Clearly  $\Delta(\cdot) = \Delta_\Phi(\cdot)$ . Denote by  $\tilde{A}_\Psi(\lambda) = (\Delta_\Psi^{jk}(\lambda))_{j,k=1}^n$  the adjugate matrix, that is,

$$A_\Psi(\lambda) \cdot \tilde{A}_\Psi(\lambda) = \tilde{A}_\Psi(\lambda) \cdot A_\Psi(\lambda) = \Delta_\Psi(\lambda) I_n, \quad (2.27)$$

and introduce the vector functions

$$U_{\Psi,j}(x, \lambda) := \sum_{k=1}^n \Delta_\Psi^{jk}(\lambda) \Psi_k(x, \lambda), \quad j \in \{1, 2, \dots, n\}, \quad (2.28)$$

$$U_\Psi(x, \lambda) := (U_{\Psi,1}(x, \lambda) \quad \dots \quad U_{\Psi,n}(x, \lambda)) = \Psi(x, \lambda) \tilde{A}_\Psi(\lambda). \quad (2.29)$$

The spectrum  $\sigma(L_{C,D})$  of the problem (1.2)–(1.4) coincides with the set of roots of the characteristic determinant  $\Delta(\cdot) = \Delta_\Phi(\cdot)$ . Assumption (ii) of Theorem 2.4 yields the relation  $\Delta(\lambda) \not\equiv 0$ . Therefore, the spectrum  $\sigma(L_{C,D})$  of the problem (1.2)–(1.4) is discrete, i.e.,  $\sigma(L_{C,D})$  consists of at most countably many eigenvalues  $\{\lambda_k\}_{k=1}^N$ ,  $N \leq \infty$ , of finite algebraic multiplicities. Let  $\lambda_k$  be an  $m_k$ -multiple zero of the function  $\Delta(\lambda)$ . As shown in the step (i) of the proof of [45, Theorem 1.2] the system of functions

$$\left\{ \frac{\partial^p}{\partial \lambda^p} U_{\Phi,j}(x, \lambda) \Big|_{\lambda=\lambda_k} : p \in \{0, 1, \dots, m_k - 1\}, j \in \{1, \dots, n\} \right\} \quad (2.30)$$

spans the root subspace  $\mathcal{R}_{\lambda_k}(L_{C,D})$  of the operator  $L_{C,D}$ , where  $U_{\Phi,j}(x, \lambda)$  is defined by the formula (2.28) for the solution  $\Phi(x, \lambda)$  in place of  $\Psi(x, \lambda)$ . Note that  $\Phi(x, \lambda)$  as well as  $U_{\Phi,j}(x, \lambda)$  and  $\Delta(\lambda)$  are entire functions of exponential type.

We prove the completeness of union of systems (2.30) for all  $k$  by contradiction. To this end, we assume that there exists a non-zero vector function  $f = \text{col}(f_1, \dots, f_n) \in L^2([0, 1]; \mathbb{C}^n)$  orthogonal to this system. Consider the entire functions

$$F_j(\lambda) := (U_{\Phi,j}(\cdot, \lambda), f(\cdot))_{L^2([0,1]; \mathbb{C}^n)}, \quad j \in \{1, \dots, n\}. \quad (2.31)$$

Since  $f$  is orthogonal to the system (2.30) then each  $\lambda_k (\in \sigma(L_{C,D}))$  is a zero of  $F_j(\cdot)$  of multiplicity at least  $m_k$ , i.e. for  $\lambda_k \in \sigma(L_{C,D})$

$$F_j^{(p)}(\lambda) \Big|_{\lambda=\lambda_k} = 0, \quad p \in \{0, 1, \dots, m_k - 1\}, \quad j \in \{1, \dots, n\}. \quad (2.32)$$

Thus, the ratio

$$G_j(\lambda) := \frac{F_j(\lambda)}{\Delta(\lambda)}, \quad j \in \{1, \dots, n\}, \quad (2.33)$$



is an entire function. Moreover, since functions  $U_{\Phi,j}(x, \lambda)$  and  $\Delta(\lambda)$  are entire functions of exponential type then so are  $G_1(\lambda), \dots, G_n(\lambda)$ . Let us prove that these functions are polynomials in  $\lambda$  by estimating their growth. Denote

$$G(\lambda) := (G_1(\lambda) \ \dots \ G_n(\lambda)). \quad (2.34)$$

It follows from (2.31) and (2.33) that

$$\int_0^1 f^*(x) U_{\Phi}(x, \lambda) dx = \Delta(\lambda) G(\lambda), \quad \lambda \in \mathbb{C}, \quad (2.35)$$

where  $f^*(x) := (\overline{f_1(x)} \ \dots \ \overline{f_n(x)}) = \overline{f(x)}^T$ .

Multiplying (2.35) by the matrix  $A_{\Phi}(\cdot)$  from the right we get in view of (2.29) and (2.27)

$$\Delta(\lambda) \int_0^1 f^*(x) \Phi(x, \lambda) dx = \Delta(\lambda) G(\lambda) A_{\Phi}(\lambda), \quad \lambda \in \mathbb{C}, \quad (2.36)$$

or equivalently

$$\int_0^1 f^*(x) \Phi(x, \lambda) dx = G(\lambda) A_{\Phi}(\lambda), \quad \lambda \notin \sigma(L_{C,D}). \quad (2.37)$$

Now the continuity of the integral in the last equality with respect to  $\lambda$ , the discreteness of the set  $\sigma(L_{C,D})$  and definition of  $A_{\Phi}(\lambda)$  (see formula (2.25)) yields the following relation

$$\int_0^1 f^*(x) \Phi(x, \lambda) dx = G(\lambda) (C\Phi(0, \lambda) + D\Phi(1, \lambda)), \quad \lambda \in \mathbb{C}. \quad (2.38)$$

Let  $\Psi(x, \lambda)$  be a fundamental  $n \times n$  matrix solution of the equation (1.2) in a domain  $S$ . Due to the initial condition  $\Phi(0, \lambda) = I_n$  the matrix functions  $\Phi(x, \lambda)$  and  $\Psi(x, \lambda)$  are related by

$$\Psi(x, \lambda) = \Phi(x, \lambda) \Psi(0, \lambda), \quad x \in [0, 1], \quad \lambda \in S, \quad (2.39)$$

where  $\Psi(0, \lambda)$  is invertible matrix function for  $\lambda \in S$ . Multiplying (2.38) by  $\Psi(0, \lambda)$  from the right we get

$$\int_0^1 f^*(x) \Psi(x, \lambda) dx = G(\lambda) (C\Psi(0, \lambda) + D\Psi(1, \lambda)), \quad \lambda \in S. \quad (2.40)$$

Now multiplying (2.40) by  $\tilde{A}_{\Psi}(x, \lambda)$  from the right we get with account of (2.29) and (2.27)

$$\int_0^1 f^*(x) U_{\Psi}(x, \lambda) dx = \Delta_{\Psi}(\lambda) G(\lambda), \quad \lambda \in S, \quad (2.41)$$

or equivalently

$$F_{\Psi,j}(\lambda) := (U_{\Psi,j}(\cdot, \lambda), f(\cdot))_{L^2([0,1]; \mathbb{C}^n)} = G_j(\lambda) \Delta_{\Psi}(\lambda), \quad \lambda \in S, \quad j \in \{1, \dots, n\}. \quad (2.42)$$

Let us estimate  $G_j(\lambda)$  from above on the rays

$$\Gamma_k := \{\lambda \in \mathbb{C} : \arg \lambda = \arg z_k\}, \quad k \in \{1, 2, 3\}, \quad (2.43)$$

using relation (2.42) for appropriate solutions  $\Psi(x, \lambda)$ .

Let  $k \in \{1, 2, 3\}$  be fixed. Since  $z_k$  is feasible then  $\Gamma_k$  lies inside some sector  $S_p$  and hence  $\Gamma_k \in S_{p,\varepsilon}$  for some  $\varepsilon > 0$ . According to Proposition 2.2 there exists a fundamental matrix solution  $Y(x, \lambda)$  of the system (1.2) with asymptotic behavior (2.12) at the domain  $S_{p,\varepsilon,R}$  for some  $R > 0$ . It was shown in the proof of [45, Theorem 1.2] that for the function  $F_{Y,j}(\lambda)$  defined by (2.42) one has

$$F_{Y,j}(\lambda) = o(e^{i\tau_k \lambda}), \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon}. \quad (2.44)$$

Next, it follows from (2.12) that

$$Y(0, \lambda) = I_n + o_n(1), \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon}, \quad (2.45)$$

where  $o_n(1)$  denotes an  $n \times n$  matrix function with entries of the form  $o(1)$  as  $\lambda \rightarrow \infty$ . Now (2.25), (2.26), (2.39) with  $Y$  in place of  $\Psi$  and (2.45) yields

$$\Delta_Y(\lambda) = \Delta(\lambda) \det Y(0, \lambda) = (1 + o(1))\Delta(\lambda), \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon}. \quad (2.46)$$

Inserting (2.44), (2.46), (2.16) into (2.42) we get

$$|G_j(\lambda)| = \left| \frac{o(e^{i\tau_k \lambda})}{(1 + o(1))\Delta(\lambda)} \right| \leq \frac{C_1 e^{\operatorname{Re}(i\tau_k \lambda)} |\lambda|^s}{C e^{\operatorname{Re}(i\tau_k \lambda)}} = C_2 |\lambda|^s, \quad \arg \lambda = \arg z_k, \quad |\lambda| > R, \quad (2.47)$$

for some  $C_1 > 0$  with  $C_2 = C_1/C$ .

Since zero is the interior point of the triangle  $\Delta_{z_1 z_2 z_3}$ , the rays  $\Gamma_1, \Gamma_2, \Gamma_3$  divide the complex plane into three closed sectors  $\Omega_1, \Omega_2, \Omega_3$  of opening less than  $\pi$ . Fix  $k \in \{1, 2, 3\}$  and apply the Phragmén-Lindelöf theorem [36, Theorem 6.1] to the function  $\tilde{G}_j(\lambda)$  considered in the sector  $\Omega_k$ . Using (2.47) we get

$$|G_j(\lambda)| \leq C_3 |\lambda|^s, \quad \lambda \in \Omega_k, \quad (2.48)$$

for some  $C_3 > 0$ , and hence

$$|G_j(\lambda)| \leq C_3 |\lambda|^s, \quad \lambda \in \mathbb{C}. \quad (2.49)$$

By the Liouville theorem (cf. [36, Theorem 1.1]),  $G_j(\lambda)$  is a polynomial of degree at most  $s$ .

Now let us prove that  $G_j(\cdot) \equiv 0$ ,  $j \in \{1, \dots, n\}$ , using equality (2.40) for appropriate solutions  $\Psi(x, \lambda)$  and the fact that  $G_j(\lambda)$  is a polynomial in  $\lambda$ . Putting (2.24) into (2.40) we get for  $k \in \{1, \dots, n\}$

$$\sum_{j=1}^n \int_0^1 \overline{f_j(x)} \psi_{jk}(x, \lambda) dx = \sum_{j=1}^n G_j(\lambda) \sum_{l=1}^n (c_{jl} \psi_{lk}(0, \lambda) + d_{jl} \psi_{lk}(1, \lambda)). \quad (2.50)$$

Consider some sector  $S_{p,\varepsilon}$ . Let  $Y(x, \lambda)$  be a matrix solution of equation (1.2) satisfying (2.12) in  $S_{p,\varepsilon,R}$ . It follows from (2.12) that

$$|y_{jk}(x, \lambda)| \leq C e^{\operatorname{Re}(ib_k \lambda)x}, \quad j, k \in \{1, \dots, n\}, \quad x \in [0, 1], \quad \lambda \in S_{p,\varepsilon,R}, \quad (2.51)$$

for some  $C > 0$ . Hence, by the Cauchy inequality,

$$\begin{aligned} \left| \sum_{j=1}^n \int_0^1 \overline{f_j(x)} y_{jk}(x, \lambda) dx \right| &\leq C \|f\|_{L^2([0,1]; \mathbb{C}^n)} \left( \int_0^1 e^{2\operatorname{Re}(ib_k \lambda)x} dx \right)^{1/2} \\ &\leq \frac{C \|f\|}{\sqrt{|\lambda|}} \max\{e^{\operatorname{Re}(ib_k \lambda)}, 1\}, \quad \lambda \in S_{p,\varepsilon,R}, \quad k \in \{1, \dots, n\}. \end{aligned} \quad (2.52)$$

Substituting (2.12) and (2.52) into (2.50) with  $Y$  in place of  $\Psi$  we get

$$\begin{aligned} &\left| \sum_{j=1}^n G_j(\lambda) \left( c_{jk} + d_{jk} e^{ib_k \lambda} + \sum_{l=1}^n (c_{jl} \cdot o(1) + d_{jl} \cdot o(1) \cdot e^{ib_k \lambda}) \right) \right| \\ &\leq \frac{C \|f\|}{\sqrt{|\lambda|}} \max\{e^{\operatorname{Re}(ib_k \lambda)}, 1\}, \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon,R}, \quad k \in \{1, \dots, n\}. \end{aligned} \quad (2.53)$$

Assume that

$$d := \max\{\deg G_j : j \in \{1, \dots, n\}\} \geq 0 \quad (2.54)$$

and denote by  $\alpha_j$  the coefficient of  $\lambda^d$  in  $G_j(\lambda)$ . So

$$G_j(\lambda) = \lambda^d(\alpha_j + o(1)), \quad \text{as } \lambda \rightarrow \infty, \quad j \in \{1, \dots, n\}. \quad (2.55)$$

From definition of  $d$  it follows that  $d = \deg G_{j_0}$  for some  $j_0 \in \{1, \dots, n\}$  and hence  $\alpha_{j_0} \neq 0$ . Therefore,  $\alpha := \text{col}(\alpha_1, \dots, \alpha_n) \neq 0$ .

Let us fix  $k \in \{1, \dots, n\}$ . Without loss of generality we may assume that

$$\text{Re}(ib_k \lambda) > 0, \quad \lambda \in S_{p,\varepsilon}. \quad (2.56)$$

Relation (2.56) yields

$$c_{jk} + d_{jk}e^{ib_k \lambda} + \sum_{l=1}^n (c_{jl} \cdot o(1) + d_{jl} \cdot o(1) \cdot e^{ib_k \lambda}) = (d_{jk} + o(1))e^{ib_k \lambda}, \quad j \in \{1, \dots, n\}, \quad (2.57)$$

as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p,\varepsilon}$ . Inserting (2.55) and (2.57) into (2.53) we get

$$|\alpha_1 d_{1k} + \dots + \alpha_n d_{nk} + o(1)| \cdot e^{\text{Re}(ib_k \lambda)} \cdot |\lambda|^d \leq \frac{C\|f\|}{\sqrt{|\lambda|}} \max\{e^{\text{Re}(ib_k \lambda)}, 1\}, \quad \lambda \in S_{p,\varepsilon,R}. \quad (2.58)$$

In view of (2.56) this estimate yields

$$\alpha_1 d_{1k} + \dots + \alpha_n d_{nk} = 0. \quad (2.59)$$

Now consider the sector  $S_{\tilde{p},\varepsilon}$  which is opposite to  $S_{p,\varepsilon}$ . In this sector due to (2.56) one has

$$\text{Re}(ib_k \lambda) < 0, \quad \lambda \in S_{\tilde{p},\varepsilon}. \quad (2.60)$$

Let  $\tilde{Y}(x, \lambda)$  be a solution of system (1.2) having asymptotic behavior (2.12) in the sector  $S_{\tilde{p},\varepsilon}$ . Inserting  $\tilde{Y}(x, \lambda)$  into (2.50) in place of  $\Psi(\cdot, \cdot)$  we get similarly to the previous case that

$$|\alpha_1 c_{1k} + \dots + \alpha_n c_{nk} + o(1)| \cdot |\lambda|^d \leq \frac{C\|f\|}{\sqrt{|\lambda|}} \max\{e^{\text{Re}(ib_k \lambda)}, 1\}, \quad \lambda \in S_{\tilde{p},\varepsilon,R}. \quad (2.61)$$

This estimate is compatible with (2.60) only if

$$\alpha_1 c_{1k} + \dots + \alpha_n c_{nk} = 0. \quad (2.62)$$

Since  $k \in \{1, \dots, n\}$  is arbitrary, combining relations (2.59) and (2.62) yields

$$D^T \alpha = 0, \quad C^T \alpha = 0, \quad (2.63)$$

which implies  $\alpha = 0$  because of the maximality condition (1.6). This contradicts the assumption  $d \geq 0$ . Hence  $G_j(\cdot) \equiv 0$  for  $j \in \{1, \dots, n\}$ .

Now it follows from (2.38) that

$$\int_0^1 \langle \Phi_j(x, \lambda), f(x) \rangle dx \equiv 0, \quad \lambda \in \mathbb{C}, \quad j \in \{1, 2, \dots, n\}. \quad (2.64)$$

By [45, Theorem 1.2, step (vi)], the vector function  $f$  satisfying (2.64) is zero, i.e. the system of root functions of the operator  $L_{C,D}(Q)$  is complete. Its minimality is implied by [45, Lemma 2.4] applied to the operator  $(L_{C,D} - \lambda)^{-1}$  with  $\lambda \in \rho(L_{C,D})$ .  $\square$

## 3. ASYMPTOTIC BEHAVIOR OF SOLUTIONS AND CHARACTERISTIC DETERMINANT

Here we refine asymptotic formulas (2.12) assuming that  $Q(\cdot)$  is continuous at the endpoints 0 and 1. These formulas will be applied to investigate asymptotic behavior of the characteristic determinant  $\Delta(\cdot)$ . We start with the following lemma.

**Lemma 3.1.** *Let  $b \in \mathbb{C} \setminus \{0\}$ ,  $C > 0$  and  $S \subset \mathbb{C}$  be a non-bounded subset of  $\mathbb{C}$  such that*

$$\operatorname{Re}(b\lambda) < -C|\lambda|, \quad \lambda \in S. \quad (3.1)$$

(i) *Let  $\varphi \in L^1[0, 1]$  and  $\varphi(\cdot)$  is continuous at zero. Then*

$$\int_0^1 e^{b\lambda t} \varphi(t) dt = \frac{\varphi(0) + o(1)}{-b\lambda}, \quad \text{as } \lambda \rightarrow \infty, \lambda \in S. \quad (3.2)$$

(ii) *Let  $\varphi \in L^1[0, 1]$  and let  $\varphi(\cdot)$  be bounded at a neighborhood of zero. Then*

$$\int_0^1 |e^{b\lambda t} \varphi(t)| dt = O(|\lambda|^{-1}), \quad \lambda \in S. \quad (3.3)$$

*Proof.* Taking into account (3.1) one has

$$\int_0^1 |e^{bt} \varphi(t)| dt \leq \left( \int_0^\delta + \int_\delta^1 \right) e^{-C|\lambda|t} |\varphi(t)| dt \leq \frac{1}{C|\lambda|} \sup_{t \in [0, \delta]} |\varphi(t)| + \|\varphi\|_1 e^{-C\delta|\lambda|}. \quad (3.4)$$

This implies (3.3). Further, (3.2) is true for  $\varphi(\cdot) \equiv \text{const}$ . Therefore, it is sufficient to prove it in the case  $\varphi(0) = 0$ . Estimate (3.4) proves this, taking into account that  $\delta$  can be chosen arbitrary small.  $\square$

Lemma 3.1 allows us to refine the asymptotic formulas (2.12) from Proposition 2.2 when  $Q$  is continuous at the endpoints of the segment  $[0, 1]$ .

**Proposition 3.2.** *Assume conditions (2.8)–(2.10) and let  $p \in \{1, \dots, \nu\}$ . Assume, in addition, that  $Q$  is continuous at the endpoints 0, 1. Then for a sufficiently large  $R$  and small  $\varepsilon > 0$  equation (1.2) has a fundamental matrix solution (2.11) analytic with respect to  $\lambda \in S_{p, \varepsilon, R}$ . Moreover,  $y_{jk}(x, \lambda)$ ,  $j, k \in \{1, \dots, n\}$ , satisfies (2.12) and has the following asymptotic behavior at the endpoints 0 and 1 as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p, \varepsilon, R}$ ,*

$$y_{jk}(0, \lambda) = \begin{cases} 0, & \text{if } \operatorname{Re}(ib_j \lambda) < \operatorname{Re}(ib_k \lambda), \\ \delta_{jk}, & \text{if } b_j = b_k, \\ \frac{b_j q_{jk}(0) + o(1)}{b_j - b_k} \cdot \frac{1}{\lambda}, & \text{if } \operatorname{Re}(ib_j \lambda) > \operatorname{Re}(ib_k \lambda); \end{cases} \quad (3.5)$$

$$y_{jk}(1, \lambda) = \begin{cases} \frac{b_j q_{jk}(1) + o(1)}{b_j - b_k} \cdot \frac{e^{ib_k \lambda}}{\lambda}, & \text{if } \operatorname{Re}(ib_j \lambda) < \operatorname{Re}(ib_k \lambda), \\ (\delta_{jk} + o(1))e^{ib_k \lambda}, & \text{if } b_j = b_k, \\ 0, & \text{if } \operatorname{Re}(ib_j \lambda) > \operatorname{Re}(ib_k \lambda). \end{cases} \quad (3.6)$$

*Proof.* According to the proof of [45, Proposition 2.2] the matrix solution  $Y(x, \lambda)$  of system (1.2) with the asymptotic behavior (2.12) in  $S_{p, \varepsilon, R}$  was constructed as the unique solution of the following system of integral equations

$$y_{jk}(x, \lambda) = \delta_{jk} e^{ib_k \lambda x} - ib_j \int_{a_{jk}}^x e^{-ib_j \lambda(t-x)} \sum_{l=1}^n q_{jl}(t) y_{lk}(t, \lambda) dt, \quad (3.7)$$

where

$$a_{jk} := \begin{cases} 0, & \text{if } \operatorname{Re}(ib_j\lambda) \leq \operatorname{Re}(ib_k\lambda), \quad \lambda \in S_{p,\varepsilon}, \\ 1, & \text{if } \operatorname{Re}(ib_j\lambda) > \operatorname{Re}(ib_k\lambda), \quad \lambda \in S_{p,\varepsilon}. \end{cases} \quad (3.8)$$

In particular,  $a_{jk} = 0$  if  $b_j = b_k$ . Let us show that this solution satisfies (3.5), (3.6). It is clear from (3.7) that for  $\lambda \in S_{p,\varepsilon,R}$  we have

$$y_{jk}(0, \lambda) = 0, \quad \operatorname{Re}(ib_j\lambda) < \operatorname{Re}(ib_k\lambda), \quad (3.9)$$

$$y_{jk}(0, \lambda) = \delta_{jk}, \quad b_j = b_k, \quad (3.10)$$

$$y_{jk}(1, \lambda) = 0, \quad \operatorname{Re}(ib_j\lambda) > \operatorname{Re}(ib_k\lambda), \quad (3.11)$$

while the second relation in (3.6) follows from Proposition 3.2. Thus, we need to prove only the third relation in (3.5) and the first one in (3.6).

At first we rewrite (2.12) in the following form

$$y_{jk}(x, \lambda) = (\delta_{jk} + \rho_{jk}(x, \lambda))e^{ib_k\lambda}, \quad j, k \in \{1, \dots, n\}, \quad (3.12)$$

where  $\rho_{jk}(x, \lambda) = o(1)$ , as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p,\varepsilon,R}$ , uniformly in  $x \in [0, 1]$ . Now inserting expression (3.12) for  $y_{jk}(x, \lambda)$  into (3.7) we obtain

$$y_{jk}(x, \lambda) = \left( \delta_{jk} - ib_j \int_{a_{jk}}^x e^{i(b_k - b_j)\lambda(t-x)} \left( q_{jk}(t) + \sum_{l=1}^n q_{jl}(t) \rho_{lk}(t, \lambda) \right) dt \right) e^{ib_k\lambda}. \quad (3.13)$$

Let  $\operatorname{Re}(ib_j\lambda) > \operatorname{Re}(ib_k\lambda)$ . Setting  $x = 0$  in (3.13) one gets

$$y_{jk}(0, \lambda) = ib_j \int_0^1 e^{i(b_k - b_j)\lambda t} q_{jk}(t) dt + ib_j \int_0^1 e^{i(b_k - b_j)\lambda t} \sum_{l=1}^n q_{jl}(t) \rho_{lk}(t, \lambda) dt. \quad (3.14)$$

Clearly,

$$\operatorname{Re}(i(b_k - b_j)\lambda) < -C|\lambda|, \quad \lambda \in S_{p,\varepsilon,R}, \quad (3.15)$$

for some  $C > 0$ . Hence, applying Lemma 3.1(i) with

$$S = S_{p,\varepsilon,R}, \quad b = i(b_k - b_j), \quad \varphi(\cdot) = ib_j q_{jk}(\cdot), \quad (3.16)$$

and taking into account the continuity of  $q_{jk}(\cdot)$  at zero, we derive from (3.2)

$$ib_j \int_0^1 e^{i(b_k - b_j)\lambda t} q_{jk}(t) dt = \frac{b_j q_{jk}(0) + o(1)}{(b_j - b_k)\lambda}, \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon,R}. \quad (3.17)$$

Further, since  $q_{jl}(\cdot)$ ,  $l \in \{1, \dots, n\}$ , is bounded at a neighborhood of zero and

$$\sup_{t \in [0,1]} |\rho_{lk}(t, \lambda)| = o(1) \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon,R}, \quad (3.18)$$

Lemma 3.1(ii) implies

$$\int_0^1 e^{i(b_k - b_j)\lambda t} \sum_{l=1}^n q_{jl}(t) \rho_{lk}(t, \lambda) dt = \sum_{l=1}^n o \left( \int_0^1 \left| e^{i(b_k - b_j)\lambda t} q_{jl}(t) \right| dt \right) = o(\lambda^{-1}), \quad (3.19)$$

as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p,\varepsilon,R}$ . This together with (3.14) and (3.17) yields the first relation in (3.5).

Next, let  $\operatorname{Re}(ib_j\lambda) < \operatorname{Re}(ib_k\lambda)$ . Then using (3.8) we obtain from (3.13)

$$y_{jk}(1, \lambda) = -ib_j e^{ib_k\lambda} \int_0^1 e^{i(b_j - b_k)\lambda s} \left( q_{jk}(1-s) + \sum_{l=1}^n q_{jl}(1-s) \rho_{lk}(1-s, \lambda) \right) ds. \quad (3.20)$$

Using the inequality

$$\operatorname{Re}(i(b_j - b_k)\lambda) < -C|\lambda|, \quad \lambda \in S_{p,\varepsilon,R}, \quad (3.21)$$

and continuity of  $q_{jl}(\cdot)$ ,  $l \in \{1, \dots, n\}$ , at the point 1, and follow the above reasoning we arrive at the third relation in (3.6).  $\square$

**Remark 3.3.** Fix  $j, k \in \{1, \dots, n\}$ . As it is clear from the proof of Proposition 3.2, the individual function  $y_{j,k}(x, \lambda)$  satisfies the third relation in (3.5) whenever  $q_{jk}(\cdot)$  is continuous at zero and  $q_{jl}(\cdot)$  is bounded at zero for  $l \in \{1, \dots, n\}$ . Otherwise it satisfies only the weaker relation

$$y_{jk}(0, \lambda) = o(1) \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon}. \quad (3.22)$$

Moreover, if  $q_{jl}(\cdot)$ ,  $l \in \{1, \dots, n\}$ , is just bounded at zero then, by Lemma 3.1(ii),  $y_{jk}(0, \lambda) = O(\lambda^{-1})$ ,  $\lambda \in S_{p,\varepsilon,R}$ . Similar statements are true for  $y_{jk}(1, \lambda)$ . This allows us to weaken assumptions on  $Q(\cdot)$  in further considerations.

In the next step we investigate the asymptotic behavior of the characteristic determinant  $\Delta(\cdot)$ . For convenience in applications we do not assume that equal  $b_j$  are grouped into blocks as it was in the previous paper [45].

**Proposition 3.4.** Let  $B$  be defined by (1.3),  $Q(\cdot) \in L^1([0, 1]; \mathbb{C}^{n \times n})$  and let  $q_{jk}$  be continuous at points 0 and 1 if  $b_j \neq b_k$ . Let, as above,  $\Delta(\cdot)$  be the characteristic determinant (2.15) of the problem (1.2)–(1.4). Finally, let  $p \in \{1, \dots, \nu\}$ . Then for sufficiently small  $\varepsilon > 0$  the characteristic determinant  $\Delta(\cdot)$  admits the following asymptotic expansion

$$\Delta(\lambda) = \gamma_p \cdot \left( \omega_0(z_p) \cdot (1 + o(1)) + \frac{\omega_1(z_p) + o(1)}{\lambda} \right) e^{i\tau_p \lambda}, \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon}. \quad (3.23)$$

Here  $z_p$  is a fixed point in  $S_{p,\varepsilon}$ ,

$$\gamma_p := \exp \left( \sum_{\operatorname{Re}(ib_j z_p) > 0} ib_j \int_0^1 q_{jj}(t) dt \right), \quad (3.24)$$

$$\tau_p := \sum_{\operatorname{Re}(ib_j z_p) > 0} b_j, \quad (3.25)$$

$$\omega_0(z_p) := \det T_{iz_p B}(C, D), \quad (3.26)$$

$$\omega_1(z_p) := \sum_{\substack{\operatorname{Re}(ib_j z_p) < 0 \\ \operatorname{Re}(ib_k z_p) > 0}} \frac{\det T_{iz_p B}^{c_j \rightarrow c_k} b_k q_{kj}(0) - \det T_{iz_p B}^{d_k \rightarrow d_j} b_j q_{jk}(1)}{b_k - b_j}, \quad (3.27)$$

and the matrix  $T_{iz_p B}^{c_j \rightarrow c_k}$  ( $T_{iz_p B}^{d_j \rightarrow d_k}$ ) is obtained from  $T_{iz_p B}(C, D)$  by replacing its  $j$ th column by the  $k$ th column of the matrix  $C$  (resp.  $D$ ).

**Remark 3.5.** Denote by  $c_j$  ( $d_j$ ) the  $j$ th column of the matrix  $C$  (resp.  $D$ ). Note that if  $\operatorname{Re}(ib_j \lambda) < 0$ , the  $j$ th column of  $T_{iz_p B}(C, D)$  coincides with  $c_j$ . Therefore, the superscript  $c_j \rightarrow c_k$  in the notation of the matrix  $T_{iz_p B}^{c_j \rightarrow c_k}$  means just replacement  $c_j$  by  $c_k$  in  $T_{iz_p B}$ . The notation  $T_{iz_p B}^{d_k \rightarrow d_j}$  is justified similarly.

*Proof of Proposition 3.4.* As in the proof of Theorem 2.4 we can assume that conditions (2.8)–(2.9) are fulfilled. Further, applying the gauge transform  $W : y \rightarrow W(x)y$  with  $W(\cdot)$  given by (2.18)–(2.19) the operator  $L_{C,D}(Q)$  is transformed into the operator  $L_{\tilde{C}, \tilde{D}}(\tilde{Q})$  with  $\tilde{D}$  and



$\tilde{Q}(x)$  given by (2.20). Due to (2.22) the characteristic determinant is preserved under this transform.

Further,  $Q - Q_1$  is continuous at the endpoints 0 and 1. Since both  $W(\cdot)$  and  $W^{-1}(\cdot)$  are continuous on  $[0, 1]$ ,  $\tilde{Q}$  is continuous at the endpoints 0 and 1 too. According to (2.20)  $\tilde{Q}$  satisfies (2.10) and, by Proposition 3.2, there exists a fundamental matrix solution  $\tilde{Y}(\cdot, \lambda)$  of system (1.2) with  $\tilde{Q}$  in place of  $Q$ , that satisfies asymptotic relations (3.5) and (3.6) with  $\tilde{q}_{jk}(\cdot)$  in place of  $q_{jk}(\cdot)$ . The fundamental matrices  $\tilde{Y}(\cdot, \lambda)$  and  $\tilde{\Phi}(\cdot, \lambda)$  are related by

$$\tilde{Y}(x, \lambda) = \tilde{\Phi}(x, \lambda)P(\lambda), \quad x \in [0, 1], \quad \lambda \in S_{p, \varepsilon, R}, \quad (3.28)$$

where  $P(\lambda) = (p_{kj}(\lambda))_{k, j=1}^n$  is an analytical invertible matrix function in  $S_{p, \varepsilon, R}$ . Hence  $\tilde{Y}(0, \lambda) = P(\lambda)$  and due to (2.12) and (2.22) (cf. [45, formula (3.31)]),

$$\Delta_{\tilde{Y}}(\lambda) := \det(C\tilde{Y}(0, \lambda) + \tilde{D}\tilde{Y}(1, \lambda)) = \tilde{\Delta}(\lambda) \det(\tilde{Y}(0, \lambda)) = (1 + o(1))\Delta(\lambda), \quad (3.29)$$

as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p, \varepsilon}$ . Thus, it suffices to prove (3.23) with  $\Delta_{\tilde{Y}}(\cdot)$  instead of  $\Delta(\cdot)$ . Since  $W(0) = I_n$ , one has  $\tilde{Q}(0) = Q(0) - Q_1(0)$  and hence

$$\tilde{Y}(0, \lambda) = Y_0 := Y_0(\lambda) := \left( y_{jk}^{[0]}(\lambda) \right)_{j, k=1}^n, \quad (3.30)$$

where  $y_{jk}^{[0]}(\lambda)$  is given by (3.5). Let us simplify  $\tilde{Y}(1, \lambda)$ . To this end let

$$\tilde{Q}(x) = \left( \tilde{Q}_{jk}(x) \right)_{j, k=1}^r, \quad \tilde{Q}_{jk}(x) \in \mathbb{C}^{n_j \times n_k}, \quad (3.31)$$

$$\tilde{Y}(x, \lambda) = \left( \tilde{Y}_{jk}(x, \lambda) \right)_{j, k=1}^r, \quad \tilde{Y}_{jk}(x, \lambda) \in \mathbb{C}^{n_j \times n_k}, \quad (3.32)$$

be the block-representations of matrices  $\tilde{Q}(x)$  and  $\tilde{Y}(x, \lambda)$  with respect to the orthogonal decomposition  $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_r}$ . It follows from (2.19)–(2.20) that

$$\tilde{Q}_{jk}(1) = W_{jj}^{-1}(1)Q_{jk}(1)W_{kk}(1), \quad j \neq k. \quad (3.33)$$

Further, note that due to (2.8)–(2.9) formula (3.6) for  $\tilde{Y}(1, \lambda)$  takes the form

$$\tilde{Y}_{jk}(1, \lambda) = \begin{cases} \frac{\beta_j \tilde{Q}_{jk}(1) + o(1)}{\beta_j - \beta_k} \cdot \frac{e^{i\beta_k \lambda}}{\lambda}, & \text{if } \operatorname{Re}(i\beta_j \lambda) < \operatorname{Re}(i\beta_k \lambda), \\ (I_{n_k} + o(1))e^{i\beta_k \lambda}, & \text{if } j = k, \\ 0, & \text{if } \operatorname{Re}(i\beta_j \lambda) > \operatorname{Re}(i\beta_k \lambda). \end{cases} \quad (3.34)$$

In view of (2.8)–(2.9) and (3.33)–(3.34) we have

$$\tilde{Y}(1, \lambda) = W^{-1}(1)Y_1W(1), \quad Y_1 := Y_1(\lambda) = \left( y_{jk}^{[1]}(\lambda) \right)_{j, k=1}^n, \quad (3.35)$$

where  $y_{jk}^{[1]}(\lambda)$  is given by (3.6). Combining (2.20), (3.29), (3.30) and (3.35) yields

$$\Delta_{\tilde{Y}}(\lambda) = \det(CY_0(\lambda) + DY_1(\lambda)W(1)) = \det(J \cdot V), \quad (3.36)$$

where

$$V := V(\lambda) := \begin{pmatrix} Y_0 \\ V_1 \end{pmatrix}, \quad V_1 := V_1(\lambda) := Y_1(\lambda)W(1), \quad \text{and} \quad J := \begin{pmatrix} C & D \end{pmatrix}. \quad (3.37)$$

By the Cauchy-Binet formula

$$\Delta_{\widehat{Y}}(\lambda) = \sum_{1 \leq k_1 < \dots < k_n \leq 2n} J \begin{pmatrix} 1 & 2 & \dots & n \\ k_1 & k_2 & \dots & k_n \end{pmatrix} \cdot V \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ 1 & 2 & \dots & n \end{pmatrix}. \quad (3.38)$$

Here  $A \begin{pmatrix} j_1 & j_2 & \dots & j_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix}$  denotes the minor of  $n \times n'$  matrix  $A = (a_{jk})$  composed of its entries located in the rows with indices  $j_1, \dots, j_p \in \{1, \dots, n\}$  and columns with indices  $k_1, \dots, k_p \in \{1, \dots, n'\}$ .

Fix a set  $\{k_1, k_2, \dots, k_n\}$  such that  $1 \leq k_1 < \dots < k_n \leq 2n$  and denote by  $m$  the number of entries of the set that do not exceed  $n$ , i.e.,

$$1 \leq k_1 < \dots < k_m \leq n < k_{m+1} < \dots < k_n. \quad (3.39)$$

Applying Laplace theorem to expand the second factor in (3.38) with respect to the first  $m$  rows, one gets

$$V \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ 1 & 2 & \dots & n \end{pmatrix} = \sum_{\substack{1 \leq j_1 < \dots < j_m \leq n \\ 1 \leq j_{m+1} < \dots < j_n < n \\ \{j_1, \dots, j_n\} = \{1, \dots, n\}}} (-1)^{(1+\dots+m)+(j_1+\dots+j_m)} \\ \times Y_0 \begin{pmatrix} k_1 & \dots & k_m \\ j_1 & \dots & j_m \end{pmatrix} \cdot V_1 \begin{pmatrix} k_{m+1} - n & \dots & k_n - n \\ j_{m+1} & \dots & j_n \end{pmatrix}. \quad (3.40)$$

It follows from (3.5) and (3.6) that

$$y_{jk}^{[0]}(\lambda) = O(1), \quad y_{jk}^{[1]}(\lambda) = O(1) \cdot e^{ib_k \lambda}, \quad \lambda \in S_{p,\varepsilon,R}, \quad j, k \in \{1, \dots, n\}. \quad (3.41)$$

Setting

$$(v_{jk}(\lambda))_{j,k=1}^n := V_1(\lambda) = Y_1(\lambda)W(1) \quad (3.42)$$

we obtain from (3.41) and the block-diagonal structure of the matrices  $B$  and  $W(1)$  that

$$v_{jk}(\lambda) = O(1) \cdot e^{ib_k \lambda}, \quad \lambda \in S_{p,\varepsilon,R}, \quad j, k \in \{1, \dots, n\}. \quad (3.43)$$

It follows from (3.30), (3.35), (3.41), (3.43) that for  $\lambda \in S_{p,\varepsilon,R}$

$$Y_0 \begin{pmatrix} k_1 & \dots & k_m \\ j_1 & \dots & j_m \end{pmatrix} = O(1), \quad (3.44)$$

$$V_1 \begin{pmatrix} k_{m+1} - n & \dots & k_n - n \\ j_{m+1} & \dots & j_n \end{pmatrix} = O(1) \cdot e^{i(b_{j_{m+1}} + \dots + b_{j_n})\lambda}. \quad (3.45)$$

Let  $\kappa$  be a number of negative values among  $\operatorname{Re}(ib_1 \lambda), \dots, \operatorname{Re}(ib_n \lambda)$ ,  $\lambda \in S_{p,\varepsilon}$ . For definiteness we assume that

$$\begin{aligned} \operatorname{Re}(ib_j \lambda) &< 0, & j &\in \{1, \dots, \kappa\}, \\ \operatorname{Re}(ib_j \lambda) &> 0, & j &\in \{\kappa + 1, \dots, n\}. \end{aligned} \quad (3.46)$$

It is clear from (3.46) that for  $\{j_{m+1}, \dots, j_n\} \neq \{\kappa + 1, \dots, n\}$  the following inequality holds

$$\operatorname{Re}(ib_{j_{m+1}} \lambda) + \dots + \operatorname{Re}(ib_{j_n} \lambda) < \operatorname{Re}(ib_{\kappa+1} \lambda) + \dots + \operatorname{Re}(ib_n \lambda) = \operatorname{Re}(i\tau_p \lambda), \quad \lambda \in S_{p,\varepsilon}, \quad (3.47)$$

where  $\tau_p$  is given by (3.25). Combining this estimate with (3.44) and (3.45) yields that for  $\{j_{m+1}, \dots, j_n\} \neq \{\kappa + 1, \dots, n\}$  and each  $h \in \mathbb{N}$ ,

$$Y_0 \begin{pmatrix} k_1 & \dots & k_m \\ j_1 & \dots & j_m \end{pmatrix} \cdot V_1 \begin{pmatrix} k_{m+1} - n & \dots & k_n - n \\ j_{m+1} & \dots & j_n \end{pmatrix} = O\left(\frac{1}{\lambda^h}\right) \cdot e^{i\tau_p \lambda}, \quad \lambda \in S_{p,\varepsilon,R}. \quad (3.48)$$

Inserting (3.48) into (3.40) we obtain for  $\lambda \in S_{p,\varepsilon,R}$  and each  $h \in \mathbb{N}$  that

$$V \begin{pmatrix} k_1 & \cdots & k_n \\ 1 & \cdots & n \end{pmatrix} = O \left( \frac{1}{\lambda^h} \right) \cdot e^{i\tau_p \lambda}, \quad m \neq \kappa; \quad (3.49)$$

$$V \begin{pmatrix} k_1 & \cdots & k_n \\ 1 & \cdots & n \end{pmatrix} = Y_0 \begin{pmatrix} k_1 & \cdots & k_\kappa \\ 1 & \cdots & \kappa \end{pmatrix} \cdot V_1 \begin{pmatrix} k_{\kappa+1} - n & \cdots & k_n - n \\ \kappa + 1 & \cdots & n \end{pmatrix} + O \left( \frac{e^{i\tau_p \lambda}}{\lambda^h} \right), \quad m = \kappa, \quad (3.50)$$

Due to the block-diagonal structure of  $W(1)$  one has

$$V_1 \begin{pmatrix} k_{\kappa+1} & \cdots & k_n \\ \kappa + 1 & \cdots & n \end{pmatrix} = Y_1 \begin{pmatrix} k_{\kappa+1} & \cdots & k_n \\ \kappa + 1 & \cdots & n \end{pmatrix} \gamma(\lambda), \quad \gamma(\lambda) := \prod_{\substack{j=1 \\ \operatorname{Re}(i\beta_j \lambda) > 0}}^r \det W_{jj}(1). \quad (3.51)$$

Applying the Liouville theorem to system (2.18) and using the definition of the sector  $S_{p,\varepsilon}$  yields

$$\gamma(\lambda) = \gamma_p, \quad \lambda \in S_{p,\varepsilon}, \quad (3.52)$$

where  $\gamma_p$  is given by (3.24). Now it follows from (3.38), (3.49), (3.50) and (3.51) that for  $\lambda \in S_{p,\varepsilon,R}$

$$\begin{aligned} \Delta_{\tilde{Y}}(\lambda) &= \gamma_p \sum_{\substack{1 \leq k_1 < \cdots < k_\kappa \leq n \\ 1 \leq k_{\kappa+1} < \cdots < k_n \leq n}} J \begin{pmatrix} 1 & \cdots & \kappa & \kappa + 1 & \cdots & n \\ k_1 & \cdots & k_\kappa & n + k_{\kappa+1} & \cdots & n + k_n \end{pmatrix} \\ &\quad \times Y_0 \begin{pmatrix} k_1 & \cdots & k_\kappa \\ 1 & \cdots & \kappa \end{pmatrix} \cdot Y_1 \begin{pmatrix} k_{\kappa+1} & \cdots & k_n \\ \kappa + 1 & \cdots & n \end{pmatrix} + O \left( \frac{1}{\lambda^h} \right) \cdot e^{i\tau_p \lambda}, \quad h \in \mathbb{N}. \end{aligned} \quad (3.53)$$

Let  $(k_1, \dots, k_\kappa) \in \mathbb{N}^\kappa$  be a sequence satisfying  $1 \leq k_1 < \dots < k_\kappa \leq n$  and let  $(l_1, \dots, l_\kappa)$  be its permutation. It is easily seen that

$$\begin{aligned} J \begin{pmatrix} 1 & \cdots & \kappa & \kappa + 1 & \cdots & n \\ k_1 & \cdots & k_\kappa & n + k_{\kappa+1} & \cdots & n + k_n \end{pmatrix} \cdot Y_0 \begin{pmatrix} k_1 & \cdots & k_\kappa \\ 1 & \cdots & \kappa \end{pmatrix} \\ = J \begin{pmatrix} 1 & \cdots & \kappa & \kappa + 1 & \cdots & n \\ l_1 & \cdots & l_\kappa & n + k_{\kappa+1} & \cdots & n + k_n \end{pmatrix} \cdot Y_0 \begin{pmatrix} l_1 & \cdots & l_\kappa \\ 1 & \cdots & \kappa \end{pmatrix}. \end{aligned} \quad (3.54)$$

This identity means that for each summand in the right-hand side of (3.53) we can choose arbitrary permutation of the corresponding sequence  $(k_1, \dots, k_\kappa)$ . Clearly, the same is true for the corresponding sequence  $(k_{\kappa+1}, \dots, k_n)$ .

It follows from (3.5) that

$$Y_0 = Y(0, \lambda) = \begin{pmatrix} I_\kappa + o(1) & O(\lambda^{-1}) \\ O(\lambda^{-1}) & I_{n-\kappa} + o(1) \end{pmatrix}, \quad \text{as } \lambda \rightarrow \infty, \quad \lambda \in S_{p,\varepsilon,R}. \quad (3.55)$$

Hence if the intersection of the sets  $\{k_1, \dots, k_\kappa\}$  and  $\{\kappa + 1, \dots, n\}$  consists of  $s$  elements, then the corresponding minor  $Y_0 \begin{pmatrix} k_1 & \cdots & k_\kappa \\ 1 & \cdots & \kappa \end{pmatrix}$  contains exactly  $s$  lines with entries of the form  $O(\lambda^{-1})$  while all entries of other lines are of the form  $O(1)$ . Indeed, if  $k_j > \kappa$ , then  $j$ th line of the considered minor coincides with the  $(k_j - \kappa)$ th line of the lower-left block of the block-matrix (3.55). Thus, we have

$$Y_0 \begin{pmatrix} k_1 & \cdots & k_\kappa \\ 1 & \cdots & \kappa \end{pmatrix} = O \left( \frac{1}{\lambda^s} \right), \quad \lambda \in S_{p,\varepsilon,R}. \quad (3.56)$$

For the cases  $s = 0$  and  $s = 1$  we can obtain sharper estimates. At first, (3.55) directly implies

$$Y_0 \begin{pmatrix} 1 & \cdots & \kappa \\ 1 & \cdots & \kappa \end{pmatrix} = 1 + o(1), \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon,R}. \quad (3.57)$$

Next, assume that  $s = 1$ , i.e. the set  $\{k_1, \dots, k_\kappa\}$  is obtained from  $\{1, \dots, \kappa\}$  by replacing its one entry by an entry from  $\{\kappa + 1, \dots, n\}$ . Assume that  $j$  is replaced by  $k$ , where  $1 \leq j \leq \kappa < k \leq n$ . Then, according to (3.46),  $\operatorname{Re}(ib_k \lambda) > 0 > \operatorname{Re}(ib_j \lambda)$  and, by (3.5),

$$\begin{aligned} Y_0 \begin{pmatrix} 1 & \cdots & j-1 & k & j+1 & \cdots & \kappa \\ 1 & \cdots & j-1 & j & j+1 & \cdots & \kappa \end{pmatrix} \\ = \det \begin{pmatrix} 1+o(1) & \cdots & o(1) & o(1) & o(1) & \cdots & o(1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ o(1) & \cdots & 1+o(1) & o(1) & o(1) & \cdots & o(1) \\ O(\lambda^{-1}) & \cdots & O(\lambda^{-1}) & \frac{r_{kj}(0)+o(1)}{\lambda} & O(\lambda^{-1}) & \cdots & O(\lambda^{-1}) \\ o(1) & \cdots & o(1) & o(1) & 1+o(1) & \cdots & o(1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ o(1) & \cdots & o(1) & o(1) & o(1) & \cdots & 1+o(1) \end{pmatrix} \\ = \frac{r_{kj}(0)+o(1)}{\lambda}, \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon,R}, \end{aligned} \quad (3.58)$$

where we set for brevity  $r_{jk}(x) := \frac{b_j q_{jk}(x)}{b_j - b_k}$ .

Further, according to (3.6)

$$Y_1 = Y(1, \lambda) = \begin{pmatrix} I_\kappa + o(1) & O(\lambda^{-1}) \\ O(\lambda^{-1}) & I_{n-\kappa} + o(1) \end{pmatrix} \cdot E(\lambda), \quad E(\lambda) := \operatorname{diag}(e^{ib_1 \lambda}, \dots, e^{ib_n \lambda}), \quad (3.59)$$

as  $\lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon,R}$ .

Let the set  $\{k_{\kappa+1}, \dots, k_n\}$  contain exactly  $s$  entries from the set  $\{1, \dots, \kappa\}$ . Then repeating the above reasoning to  $Y_1$  in place of  $Y_0$  yields

$$Y_1 \begin{pmatrix} k_{\kappa+1} & \cdots & k_n \\ \kappa+1 & \cdots & n \end{pmatrix} = O\left(\frac{1}{\lambda^s}\right) e^{i\tau_p \lambda}, \quad \lambda \in S_{p,\varepsilon,R}. \quad (3.60)$$

Further, it is easily seen that

$$Y_1 \begin{pmatrix} \kappa+1 & \cdots & n \\ \kappa+1 & \cdots & n \end{pmatrix} = (1 + o(1)) \cdot e^{i\tau_p \lambda}; \quad (3.61)$$

$$Y_1 \begin{pmatrix} \kappa+1 & \cdots & k-1 & j & k+1 & \cdots & n \\ \kappa+1 & \cdots & k-1 & k & k+1 & \cdots & n \end{pmatrix} = (r_{jk}(1) + o(1)) \cdot \frac{e^{i\tau_p \lambda}}{\lambda}, \quad (3.62)$$

as  $\lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon,R}$ , where  $j \in \{1, \dots, \kappa\}$  and  $k \in \{\kappa + 1, \dots, n\}$ .

Inserting formulas (3.56) and (3.60) into (3.53) and using (3.54) we get

$$\begin{aligned}
\gamma_p^{-1} \Delta_{\tilde{Y}}(\lambda) = & \left( J \begin{pmatrix} 1 & \dots & \kappa & \kappa+1 & \dots & n \\ 1 & \dots & \kappa & n+\kappa+1 & \dots & n+n \end{pmatrix} \cdot Y_0 \begin{pmatrix} 1 & \dots & \kappa \\ 1 & \dots & \kappa \end{pmatrix} \cdot Y_1 \begin{pmatrix} \kappa+1 & \dots & n \\ \kappa+1 & \dots & n \end{pmatrix} \right. \\
& + \sum_{j=1}^{\kappa} \sum_{k=\kappa+1}^n J \begin{pmatrix} 1 & \dots & j-1 & j & j+1 & \dots & \kappa & \kappa+1 & \dots & n \\ 1 & \dots & j-1 & k & j+1 & \dots & \kappa & n+\kappa+1 & \dots & n+n \end{pmatrix} \cdot Y_0 \begin{pmatrix} 1 & \dots & j-1 & k & j+1 & \dots & \kappa \\ 1 & \dots & j-1 & j & j+1 & \dots & \kappa \end{pmatrix} \cdot Y_1 \begin{pmatrix} \kappa+1 & \dots & n \\ \kappa+1 & \dots & n \end{pmatrix} \\
& + \sum_{j=1}^{\kappa} \sum_{k=\kappa+1}^n J \begin{pmatrix} 1 & \dots & \kappa & \kappa+1 & \dots & k-1 & k & k+1 & \dots & n \\ 1 & \dots & \kappa & n+\kappa+1 & \dots & n+k-1 & n+j & n+k+1 & \dots & n+n \end{pmatrix} \\
& \times Y_0 \begin{pmatrix} 1 & \dots & \kappa \\ 1 & \dots & \kappa \end{pmatrix} \cdot Y_1 \begin{pmatrix} \kappa+1 & \dots & k-1 & j & k+1 & \dots & n \\ \kappa+1 & \dots & k-1 & k & k+1 & \dots & n \end{pmatrix} \Big) + O\left(\frac{1}{\lambda^2}\right) e^{i\tau_p \lambda}, \quad \lambda \in S_{p,\varepsilon,R}. \tag{3.63}
\end{aligned}$$

Let  $z_p$  be some fixed point in  $S_{p,\varepsilon}$ . Then it is clear from inequalities (3.46) and definition of matrices  $T_{iz_p B}(C, D)$ ,  $T_{iz_p B}^{c_j \rightarrow c_k}$  and  $T_{iz_p B}^{d_k \rightarrow d_j}$  that

$$J \begin{pmatrix} 1 & \dots & \kappa & \kappa+1 & \dots & n \\ 1 & \dots & \kappa & n+\kappa+1 & \dots & n+n \end{pmatrix} = \det T_{iz_p B}(C, D), \tag{3.64}$$

$$J \begin{pmatrix} 1 & \dots & j-1 & j & j+1 & \dots & \kappa & \kappa+1 & \dots & n \\ 1 & \dots & j-1 & k & j+1 & \dots & \kappa & n+\kappa+1 & \dots & n+n \end{pmatrix} = \det T_{iz_p B}^{c_j \rightarrow c_k}, \tag{3.65}$$

$$J \begin{pmatrix} 1 & \dots & \kappa & \kappa+1 & \dots & k-1 & k & k+1 & \dots & n \\ 1 & \dots & \kappa & n+\kappa+1 & \dots & n+k-1 & n+j & n+k+1 & \dots & n+n \end{pmatrix} = \det T_{iz_p B}^{d_k \rightarrow d_j}. \tag{3.66}$$

Now inserting (3.57), (3.58), (3.61), (3.62) and (3.64), (3.65), (3.66) into (3.63) we get

$$\begin{aligned}
\gamma_p^{-1} \Delta_{\tilde{Y}}(\lambda) = & \det T_{iz_p B}(C, D) \cdot (1 + o(1)) \cdot (1 + o(1)) \cdot e^{i\tau_p \lambda} \\
& + \sum_{j=1}^{\kappa} \sum_{k=\kappa+1}^n \det T_{iz_p B}^{c_j \rightarrow c_k} \cdot \frac{r_{kj}(0) + o(1)}{\lambda} \cdot (1 + o(1)) \cdot e^{i\tau_p \lambda} \\
& + \sum_{j=1}^{\kappa} \sum_{k=\kappa+1}^n \det T_{iz_p B}^{d_k \rightarrow d_j} \cdot (1 + o(1)) \cdot \frac{r_{jk}(1) + o(1)}{\lambda} \cdot e^{i\tau_p \lambda} + O\left(\frac{1}{\lambda^2}\right) e^{i\tau_p \lambda} \tag{3.67} \\
= & e^{i\tau_p \lambda} \cdot \left( \omega_0(z_p) \cdot (1 + o(1)) + o(\lambda^{-1}) \right. \\
& \left. + \sum_{j=1}^{\kappa} \sum_{k=\kappa+1}^n \frac{\det T_{iz_p B}^{c_j \rightarrow c_k} b_k q_{kj}(0) - \det T_{iz_p B}^{d_k \rightarrow d_j} b_j q_{jk}(1)}{\lambda(b_k - b_j)} \right), \tag{3.68}
\end{aligned}$$

as  $\lambda \rightarrow \infty$ ,  $\lambda \in S_{p,\varepsilon,R}$ . Rewriting the double sum in the last equality with account of (3.46) we arrive at formula (3.23) with the required form of  $\omega_1(z_p)$ .  $\square$

Next we present an asymptotic formula for the characteristic determinant  $\Delta(\cdot)$  which will be needed in the sequel. It can be obtained by repeating the proof of Proposition 3.4 but using Proposition 2.2 in place of Proposition 3.2 for estimating the solution  $Y(x, \lambda)$ .

**Lemma 3.6.** *Assume that  $Q(\cdot) \in L^1([0, 1]; \mathbb{C}^{n \times n})$ . Let  $p \in \{1, \dots, \nu\}$ . Then for sufficiently small  $\varepsilon > 0$  the characteristic determinant  $\Delta(\cdot)$  admits the following asymptotic behavior*

$$\Delta(\lambda) = \gamma_p \cdot (\det T_{iz_p B}(C, D) + o(1)) e^{i\tau_p \lambda}, \quad \text{as } \lambda \rightarrow \infty, \lambda \in S_{p,\varepsilon}. \tag{3.69}$$

Here  $z_p$  is a fixed point in  $S_{p,\varepsilon}$ , while  $\gamma_p$  and  $\tau_p$  are given by (3.24) and (3.25), respectively.

Formula (3.69) can also be extracted from the proof of [45, Theorem 1.2] (cf. formula (3.38) from [45]).

## 4. EXPLICIT COMPLETENESS RESULTS

**4.1. Explicit sufficient conditions of completeness.** Now we are ready to state our main result on completeness of the root vectors of the boundary value problem (1.2)–(1.4) in terms of the matrices  $B, C, D$  and  $Q(\cdot)$ .

**Theorem 4.1.** *Assume that  $Q(\cdot) \in L^1([0, 1]; \mathbb{C}^{n \times n})$  and  $q_{jk}$  is continuous at points 0 and 1 if  $b_j \neq b_k$ . Let  $\omega_0(z_k)$  and  $\omega_1(z_k)$  be given by (3.26) and (3.27), respectively. Assume also that there exist three admissible complex numbers  $z_1, z_2, z_3$  satisfying the following conditions:*

- (a) *the origin is an interior point of the triangle  $\Delta_{z_1 z_2 z_3}$ ;*
- (b)  *$|\omega_0(z_k)| + |\omega_1(z_k)| \neq 0$ ,  $k \in \{1, 2, 3\}$ .*

*Then the system of root functions of the BVP (1.2)–(1.4) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .*

**Remark 4.2.** *Note that  $\omega_j(\cdot)$ ,  $j \in \{0, 1\}$ , is a constant function in each sector  $\sigma_k$ ,  $k \in \{1, \dots, m\}$ , introduced before formula (1.8). Hence  $\omega_j(\cdot)$ ,  $j \in \{0, 1\}$ , is piecewise constant function in the plane  $\mathbb{C}$  with cuts along the lines  $\partial\sigma_k$ ,  $k \in \{1, \dots, m\}$ . It is easily seen that the assumptions of Theorem 4.1 fail if and only if both  $\omega_0(\cdot)$  and  $\omega_1(\cdot)$  vanish in the open half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re}(c\lambda) > 0\}$  for some  $c \neq 0$ .*

*Proof of Theorem 4.1.* Recall that the lines  $l_j = \{\lambda \in \mathbb{C} : \operatorname{Re}(ib_j\lambda) = 0\}$  divide the complex plane into  $m$  sectors  $\sigma_1, \dots, \sigma_m$ . Let  $k \in \{1, 2, 3\}$  be fixed. Note that the point  $z_k$  can be not feasible but it is clear from definition of  $\omega_0(\cdot)$  and  $\omega_1(\cdot)$  that they are constant in each sector  $\sigma_j$ . Hence if  $z_k$  is not feasible, that is, it lies at one of the lines  $l_{jk} = \{\lambda \in \mathbb{C} : \operatorname{Re}(ib_j\lambda) = \operatorname{Re}(ib_k\lambda)\}$ , we can replace it by any point with arbitrary close argument to make it feasible and to conserve the condition (a) of the theorem. Thus, we can assume that the points  $z_1, z_2, z_3$  are feasible. Then combining condition (b) of the theorem with Proposition 3.4 implies for  $k \in \{1, 2, 3\}$

$$|\Delta(\lambda)| \geq C \left| \omega_0(z_k) + \frac{\omega_1(z_k)}{\lambda} \right| e^{\operatorname{Re}(i\tau_k\lambda)} \geq C_1 \frac{e^{\operatorname{Re}(i\tau_k\lambda)}}{|\lambda|}, \quad |\lambda| > R, \quad \arg \lambda = \arg z_k, \quad (4.1)$$

where  $C, C_1 > 0$ ,  $\tau_k := \sum_{\operatorname{Re}(ib_j z_k) > 0} b_j$  and  $R$  is sufficiently large. To complete the proof it remains to apply Theorem 2.4.  $\square$

The following result is easily derived from Theorem 4.1 (cf. [45, Corollary 3.2]).

**Corollary 4.3.** *Let  $Q$  satisfy assumptions of Theorem 4.1, and let  $|\omega_0(\pm z)| + |\omega_1(\pm z)| \neq 0$  for some admissible number  $z$ . Then the system of root functions of the BVP (1.2)–(1.4) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .*

**Remark 4.4.** *In connection with Theorem 4.1 we mention the fundamental paper [59] by A.A. Shkalikov, where he studied BVP for ODE (1.1) with spectral parameter in boundary conditions. In particular, the notion of  $B$ -weakly regular boundary conditions might be treated as an analogue of the notion of normal BVP of order 0 from [59], while conditions of Theorem 4.1 correlate with those of normal BVP of order 1 from [59]. Moreover, it is proved in [59] that the system of root functions of the linearization of the normal BVP for ODE (1.1) is complete in certain direct sums of Sobolev spaces. For certain matrices  $B = \operatorname{diag}(b_1, \dots, b_n)$  with simple spectrum this result correlate with [45, Theorem 1.2] and Theorem 4.1.*

We first apply Theorem 4.1 to  $2 \times 2$  case. Let

$$\begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \quad J_{jk} := \det \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}, \quad j, k \in \{1, \dots, 4\}. \quad (4.2)$$



**Proposition 4.5.** *Let  $n = 2$ ,  $\arg b_1 \neq \arg b_2$ , and let  $q_{12}, q_{21}$  be continuous at the endpoints 0 and 1. Then the system of root functions of the boundary value problem (1.2)–(1.4) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^2)$  whenever*

$$|J_{32}| + |b_1 J_{13} q_{12}(0) + b_2 J_{42} q_{21}(1)| \neq 0, \quad (4.3)$$

$$|J_{14}| + |b_1 J_{13} q_{12}(1) + b_2 J_{42} q_{21}(0)| \neq 0. \quad (4.4)$$

*Proof.* Since  $\arg b_1 \neq \arg b_2$  then there exists  $z \in \mathbb{C}$  such that  $\operatorname{Re}(ib_1 z) < 0 < \operatorname{Re}(ib_2 z)$ . Then, in accordance with definition of  $J_{jk}$  and the numbers  $\omega_0(z), \omega_1(z)$ ,

$$\omega_0(z) = J_{14}, \quad \omega_1(z) = \frac{J_{24} b_1 q_{21}(0) - J_{13} b_1 q_{12}(1)}{b_1 - b_2}, \quad (4.5)$$

$$\omega_0(-z) = J_{32}, \quad \omega_1(-z) = \frac{J_{31} b_2 q_{12}(0) - J_{42} b_2 q_{21}(1)}{b_2 - b_1}. \quad (4.6)$$

Conditions (4.3), (4.4) imply  $|\omega_0(\pm z)| + |\omega_1(\pm z)| \neq 0$ . Hence Corollary 4.3 yields the result.  $\square$

**Remark 4.6.** *In the case of  $2 \times 2$  Dirac-type systems ( $b_1 < 0 < b_2$ ) this result improves Theorem 5.1 from [45] where the completeness was proved under the stronger assumption  $q_{12}, q_{21} \in C^1[0, 1]$  while was stated for  $q_{12}, q_{21} \in C[0, 1]$ . It happened because the precise version of Lemma 5.4 from [45] requires a stronger assumption  $Q(\cdot) \in C^1([0, 1]; \mathbb{C}^{n \times n})$  instead of  $Q(\cdot) \in C([0, 1]; \mathbb{C}^{n \times n})$  (cf. [41, Theorem 1.1]). In our forthcoming paper the completeness property of BVP (1.2)–(1.4) for  $2 \times 2$  Dirac-type systems will be discussed in detail.*

*In the case  $b_2 b_1^{-1} \notin \mathbb{R}$  Proposition 4.5 improves Theorems 1.4 and 1.6 from [1] where the completeness property was proved for analytic  $Q(\cdot)$ .*

The next result demonstrates that Theorem 4.1 cannot be treated as a perturbation result since unperturbed operator  $L_{C,D}(0)$  may have incomplete system of root functions.

**Corollary 4.7.** *Let  $\kappa \in \{1, \dots, n-1\}$ ,  $\operatorname{Re} b_j < 0$  for  $j \in \{1, \dots, \kappa\}$ ,  $\operatorname{Re} b_j > 0$  for  $j \in \{\kappa+1, \dots, n\}$ , and the first boundary condition in (1.4) is of the form  $y_1(0) = 0$ . Then the following holds:*

(i) *Assume that  $Q$  is continuous at the endpoints 0 and 1 of the segment  $[0, 1]$ ,*

$$\det T_B(C, D) \neq 0 \quad \text{and} \quad \sum_{j=\kappa+1}^n \frac{\det T_{-B}^{c_j \rightarrow c_1}}{b_1 - b_j} \cdot q_{1j}(0) \neq 0. \quad (4.7)$$

*Then the system of root functions of the operator  $L_{C,D}(Q)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .*

(ii) *If  $q_{1j}(x) = 0$  for  $x \in [0, \varepsilon]$ ,  $j \in \{2, \dots, n\}$ , for some  $\varepsilon > 0$ , then the system of root functions of the operator  $L_{C,D}(Q)$  is incomplete in  $L^2([0, 1]; \mathbb{C}^n)$  and its defect is infinite. In particular, the latter is valid for the operator  $L_{C,D}(0)$  with zero potential.*

*Proof.* (i) The condition  $y_1(0) = 0$  means that  $c_{11} = 1$ ,  $c_{1k} = 0$  for  $k \in \{2, \dots, n\}$ , and  $d_{1k} = 0$ ,  $k \in \{1, \dots, n\}$ . Therefore, the matrix  $T_{-B}(C, D)$  has zero first line and hence  $\omega_0(i) = 0$ . Moreover, due to the structure of the first row of  $(C \ D)$ ,  $\det T_{-B}^{d_k \rightarrow d_j} = 0$ ,  $j, k \in \{1, \dots, n\}$ , and  $\det T_{-B}^{c_j \rightarrow c_k} = 0$ , for  $k > 1$ . Now the assumption on  $\operatorname{Re} b_j$ , definition of  $\omega_1(\cdot)$ , and condition (4.7) together imply

$$\omega_1(i) = \sum_{j=\kappa+1}^n \frac{\det T_{-B}^{c_j \rightarrow c_1} \cdot b_1 q_{1j}(0)}{b_1 - b_j} \neq 0. \quad (4.8)$$

Due to the first relation in (4.7)  $\omega_0(-i) = \det T_B(C, D) \neq 0$ . It remains to apply Corollary 4.3.

(ii) Under our assumption each solution  $y = \text{col}(y_1, \dots, y_n)$  of the problem (1.2)–(1.4) satisfies

$$y'_1 = ib_1 \lambda y_1 + ib_1 q_{11}(x) y_1, \quad x \in [0, \varepsilon], \quad \text{and} \quad y_1(0) = 0. \quad (4.9)$$

By the uniqueness theorem,  $y_1(x) = 0$  for  $x \in [0, \varepsilon]$ . Hence each  $f = \text{col}(f_1, 0, \dots, 0) \in L^2([0, 1]; \mathbb{C}^n)$  with  $f_1$  vanishing on  $[\varepsilon, 1]$  is orthogonal to the system of root functions of the operator  $L_{C,D}(Q)$ .  $\square$

**Remark 4.8.** Let  $n = 3$ ,  $\kappa = 1$  and  $y_1(0) = 0$ . Then condition (4.7) takes the form

$$\begin{vmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} d_{21} & c_{23} \\ d_{31} & c_{33} \end{vmatrix} \cdot \frac{q_{12}(0)}{b_2 - b_1} + \begin{vmatrix} d_{21} & c_{22} \\ d_{31} & c_{32} \end{vmatrix} \cdot \frac{q_{13}(0)}{b_1 - b_3} \neq 0. \quad (4.10)$$

Therefore, if  $|q_{12}(0)| + |q_{13}(0)| \neq 0$ , then, in general, the system of root functions of the operator  $L_{C,D}(Q)$  with the first boundary condition  $y_1(0) = 0$ , is complete in  $L^2([0, 1]; \mathbb{C}^3)$ .

Finally, we specify Corollary 4.3 for  $4 \times 4$  Dirac type equation subject to special boundary conditions. This statement will be applied in Section 7 for study of the Timoshenko beam model.

**Corollary 4.9.** Let  $n = 4$ ,  $B = \text{diag}(-b_1, b_1, -b_2, b_2)$ , where  $b_1, b_2 > 0$ , let  $Q \in L^1([0, 1]; \mathbb{C}^{4 \times 4})$ , where  $Q$  is continuous at the endpoints 0 and 1, and matrices  $C$  and  $D$  are of the form

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & d_4 \end{pmatrix}. \quad (4.11)$$

Assume that

$$|d_2 d_4| + |d_1 d_4 q_{12}(1)| + |d_2 d_3 q_{34}(1)| \neq 0, \quad |d_1 d_3| + |d_2 d_3 q_{21}(1)| + |d_1 d_4 q_{43}(1)| \neq 0. \quad (4.12)$$

Then the system of root functions of the BVP (1.2)–(1.4) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^4)$ .

*Proof.* By the definition of the matrix  $T_B(C, D)$ ,

$$T_B(C, D) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}, \quad (4.13)$$

and hence

$$\omega_0(-i) = \det T_B(C, D) = d_2 d_4. \quad (4.14)$$

In our case the double sum in (3.27) for  $\omega_1(-i)$  involves only values  $j = 1, 3$  and  $k = 2, 4$ . It follows from definition of matrices  $T_{izB}^{c_j \rightarrow c_k}$  and  $T_{izB}^{d_k \rightarrow d_j}$  that

$$\det T_B^{c_1 \rightarrow c_2} = d_2 d_4, \quad \det T_B^{d_2 \rightarrow d_1} = d_1 d_4, \quad (4.15)$$

$$\det T_B^{c_1 \rightarrow c_4} = 0, \quad \det T_B^{d_4 \rightarrow d_1} = 0, \quad (4.16)$$

$$\det T_B^{c_3 \rightarrow c_2} = 0, \quad \det T_B^{d_2 \rightarrow d_3} = 0, \quad (4.17)$$

$$\det T_B^{c_3 \rightarrow c_4} = d_2 d_4, \quad \det T_B^{d_4 \rightarrow d_3} = d_2 d_3. \quad (4.18)$$

Inserting these expressions into (3.27) we obtain

$$\omega_1(-i) = \frac{1}{2} (d_2 d_4 q_{21}(0) + d_1 d_4 q_{12}(1) + d_2 d_4 q_{43}(0) + d_2 d_3 q_{34}(1)). \quad (4.19)$$

Note that if  $d_2 = 0$ , then  $\omega_1(-i) = \frac{1}{2}d_1d_4q_{12}(1)$ . On the other hand, if  $d_4 = 0$ , then  $\omega_1(-i) = \frac{1}{2}d_2d_3q_{34}(1)$ . This allows us to rewrite condition  $|\omega_0(-i)| + |\omega_1(-i)| \neq 0$  in the form of the first relation in (4.12).

Similarly, one verifies that condition  $|\omega_0(i)| + |\omega_1(i)| \neq 0$  turns into the second relation in (4.12). One completes the proof by applying Corollary 4.3.  $\square$

The following simple lemma will be useful for us in Section 6.

**Lemma 4.10.** *Condition (4.12) is fulfilled if and only if each of the following conditions is satisfied*

$$|d_1| + |d_2| \neq 0, \quad |d_3| + |d_4| \neq 0, \quad (4.20)$$

$$|d_1| + |d_3| \neq 0, \quad |d_2| + |d_4| \neq 0, \quad (4.21)$$

$$|d_1| + |q_{21}(1)| \neq 0, \quad |d_2| + |q_{12}(1)| \neq 0, \quad (4.22)$$

$$|d_3| + |q_{43}(1)| \neq 0, \quad |d_4| + |q_{34}(1)| \neq 0. \quad (4.23)$$

*Proof.* If  $d_1d_2d_3d_4 \neq 0$  then the statement is obvious. Further assume that  $d_j = 0$  for some  $j \in \{1, 2, 3, 4\}$ . Let for definiteness,  $d_1 = 0$ . Then conditions (4.20)–(4.23) are satisfied if and only if

$$d_2d_3q_{21}(1) \neq 0 \quad \text{and} \quad |d_4| + |q_{34}(1)| \neq 0. \quad (4.24)$$

This, in turn, is equivalent to (4.12) whenever  $d_1 = 0$ , and we are done.  $\square$

**4.2. Example.** Here we illustrate Proposition 4.5 by investigation of the completeness and minimality of the system of vector functions

$$\left\{ \begin{pmatrix} e^{anx} \sin nx \\ ne^{anx} (\sin nx + i \cos nx) \end{pmatrix} \right\}_{n \in \mathbb{Z} \setminus \{0\}}, \quad a \in \mathbb{C}, \quad (4.25)$$

in the space  $L^2([0, \pi]; \mathbb{C}^2)$ .

**Corollary 4.11.** *Let*

$$ia \notin (-\infty, -1] \cap [1, \infty). \quad (4.26)$$

*Then system (4.25) is complete and minimal in  $L^2([0, \pi]; \mathbb{C}^2)$ .*

*Proof.* Since  $a \neq \pm i$  there exists  $\theta \in \mathbb{C} \setminus \{\pi n\}_{n \in \mathbb{Z}}$  such that  $a = \operatorname{ctg} \theta$ . Consider the following boundary value problem

$$\begin{cases} y_1' = e^{i\theta} \lambda y_1 + y_2, \\ y_2' = e^{-i\theta} \lambda y_2, \end{cases} \quad (4.27)$$

$$y_1(0) = y_1(1) = 0. \quad (4.28)$$

Straightforward calculation shows that its spectrum is simple, consists of the eigenvalues  $\left\{ \frac{\pi n}{\sin \theta} \right\}_{n \in \mathbb{Z} \setminus \{0\}}$ , and the system of the corresponding eigenfunctions is

$$\left\{ \begin{pmatrix} e^{a\pi nx} \sin \pi nx \\ \pi n \cdot e^{(a-i)\pi nx} \end{pmatrix} \right\}_{n \in \mathbb{Z} \setminus \{0\}}. \quad (4.29)$$

It is easily seen that a potential matrix of the operator  $L_{C,D}(Q)$  associated with the boundary value problem (4.27)–(4.28) is constant:  $Q(\cdot) = \begin{pmatrix} 0 & -e^{-i\theta} \\ 0 & 0 \end{pmatrix}$ . Clearly,

$$B = \operatorname{diag}(b_1, b_2) := -i \operatorname{diag}(e^{i\theta}, e^{-i\theta}). \quad (4.30)$$

Moreover, due to (4.26)  $\arg b_1 \neq \arg b_2$ .

Clearly, boundary conditions (4.28) imply  $J_{13} = 1$ , while the other determinants  $J_{jk}$  are zero. In particular, boundary conditions (4.28) are non-weakly regular and even degenerate:  $\Delta_0(\cdot) \equiv 0$ . However, conditions (4.3)–(4.4) take now the form  $q_{12}(0)q_{12}(1) \neq 0$  and clearly, are fulfilled. Hence, by Proposition 4.5, the system of eigenvectors (4.29) is complete and minimal in  $L^2([0, 1]; \mathbb{C}^2)$ . The latter is equivalent to the completeness and minimality of the system (4.25) in  $L^2([0, \pi]; \mathbb{C}^2)$ .  $\square$

**Remark 4.12.** *In connection with Corollary 4.11 let us consider one more system of functions  $\mathcal{K}_a = \{e^{anx} \sin nx\}_{n \in \mathbb{Z} \setminus \{0\}}$ . Clearly, it is a system of the eigenfunctions of the problem*

$$y'' - 2a\lambda y' + (a^2 + 1)\lambda^2 y = 0, \quad y(0) = y(\pi) = 0. \quad (4.31)$$

*It is known (see [36, Part II, Appendix A1], [39] and the references therein) that this system is twofold complete in  $L^2[0, \pi]$  in the sense of M.V. Keldysh [30]. The latter means completeness of the system*

$$\{\operatorname{col}(e^{anx} \sin nx, ne^{anx} \sin nx)\}_{n \in \mathbb{Z} \setminus \{0\}} \quad (4.32)$$

*in  $L^2([0, \pi]; \mathbb{C}^2)$ . So, the statement of Corollary 4.11 is in a sense close to the twofold completeness and minimality of the system  $\mathcal{K}_a$ . Note that investigation of the completeness and basis property of a "half" system  $\mathcal{K}_a^+ := \{e^{anx} \sin nx\}_{n=1}^\infty$  in  $L^2[0, \pi]$  has been initiated by A.G. Kostyuchenko and constitutes his named problem.*

*Note also that in the case  $a \in \mathbb{R}$  problem (4.31) naturally arises in the investigation of the solvability of the following elliptic boundary value problem in the strip  $\Omega = [0, \pi] \times \mathbb{R}_+$ :*

$$\begin{cases} Lu := \frac{\partial^2 u}{\partial x^2} - 2a \frac{\partial^2 u}{\partial x \partial t} + (a^2 + 1) \frac{\partial^2 u}{\partial t^2} = 0, \\ u(0, t) = u(\pi, t) = 0, \quad t \geq 0, \\ u(x, 0) = u_0(x), \quad u_0 \in L^2[0, \pi]. \end{cases} \quad (4.33)$$

*Since equation  $Lu = 0$  is elliptic, the Cauchy problem in the strip is incorrect. Applying the Fourier method, i.e. seeking for a solution of (4.33) in the form  $u(x, t) = e^{\lambda t} y(x)$ , leads to problem (4.31).*

**4.3. Necessary conditions of completeness.** Next we present some necessary conditions of completeness.

**Proposition 4.13.** *Let boundary conditions (1.4) be of the form  $y(0) = Ay(1)$ , where  $\det A \neq 0$ ,*

$$AB + BA = 0 \quad \text{and} \quad Q(1 - x) = A^{-1}Q(x)A, \quad x \in [0, \varepsilon], \quad \text{for some } \varepsilon > 0. \quad (4.34)$$

*Then the defect of the system of root functions of the operator  $L_{C,D}(Q)$  in  $L^2([0, 1]; \mathbb{C}^n)$  is infinite.*

*Proof.* Let  $\lambda$  be an eigenvalue of  $L$  and let  $\{u_p(x)\}_{p=1}^m$  be a chain of the eigenfunction and associated functions of the operator  $L$  corresponding to  $\lambda$ . Put  $u_0(x) := 0$ . It is clear that  $u_p(\cdot)$ ,  $p \in \{0, 1, \dots, m\}$ , satisfies boundary conditions (1.4) and the following identity holds

$$Lu_p(x) = \lambda u_p(x) + u_{p-1}(x), \quad x \in [0, 1], \quad p \in \{1, \dots, m\}. \quad (4.35)$$

Denote  $v_p(x) := Au_p(1 - x)$ . It follows from (4.35) and (1.2) that

$$\begin{aligned} (u'_p)(1 - x) &= iB(\lambda - Q(1 - x))u_p(1 - x) + iBu_{p-1}(1 - x) \\ &= iB[(\lambda - Q(1 - x))A^{-1}v_p(x) + A^{-1}v_{p-1}(x)]. \end{aligned} \quad (4.36)$$

Further, combining relations (4.34) with the definition of  $v_p$  yields

$$\begin{aligned}
Lv_p(x) &= -iB^{-1}v'_p(x) + Q(x)v_p(x) \\
&= iB^{-1}A \cdot (u'_p)(1-x) + Q(x)v_p(x) \\
&= -iAB^{-1}iB [(\lambda - Q(1-x))A^{-1}v_p(x) + A^{-1}v_{p-1}(x)] + Q(x)v_p(x) \\
&= \lambda v_p(x) + v_{p-1}(x) + (Q(x) - AQ(1-x)A^{-1})v_p(x) \\
&= \lambda v_p(x) + v_{p-1}(x), \quad x \in [0, \varepsilon].
\end{aligned} \tag{4.37}$$

Next, due to the assumption,  $v_p(0) = Au_p(1) = u_p(0)$ . Thus, both  $u_p$  and  $v_p$  satisfy the same equation (4.35) for  $x \in [0, \varepsilon]$  and have equal initial data at zero. Therefore, by the Cauchy uniqueness theorem,

$$u_p(x) = v_p(x) = Au_p(1-x), \quad x \in [0, \varepsilon]. \tag{4.38}$$

Further, let  $f \in L^2([0, 1]; \mathbb{C}^n)$  and let

$$f(x) = 0 \quad \text{for } x \in [\varepsilon, 1-\varepsilon] \quad \text{and} \quad f(1-x) = -A^*f(x), \quad \text{for } x \in [0, \varepsilon]. \tag{4.39}$$

Then one has for  $p \geq 0$

$$\begin{aligned}
\int_0^1 \langle u_p(x), f(x) \rangle dx &= \int_0^\varepsilon \langle u_p(x), f(x) \rangle dx + \int_0^\varepsilon \langle u_p(1-x), f(1-x) \rangle dx \\
&= \int_0^\varepsilon (\langle u_p(x), f(x) \rangle + \langle A^{-1}u_p(x), -A^*f(x) \rangle) dx = 0.
\end{aligned} \tag{4.40}$$

This identity shows that each vector-function  $f$  satisfying (4.39) is orthogonal to the system of root functions of the operator  $L_{C,D}(Q)$ . This completes the proof.  $\square$

Note that existence of a nonsingular solution of the matrix equation  $AB+BA=0$  is equivalent to the similarity of the matrices  $B$  and  $-B$ :  $ABA^{-1} = -B$ . Since  $B$  is diagonal, the latter amounts to saying that the spectra  $\sigma(B)$  and  $\sigma(-B)$  coincide with their multiplicities. Thus we can restate Proposition 4.13 as follows.

**Corollary 4.14.** *Let  $n = 2p$  and  $B = \text{diag}(\tilde{B}, -\tilde{B})$ , where*

$$\tilde{B} = \text{diag}(I_{n_1}b_1, \dots, I_{n_r}b_r), \quad n_1 + \dots + n_r = p. \tag{4.41}$$

Further, let

$$A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, \quad A_j = \text{diag}(A_{j1}, \dots, A_{jr}), \quad A_{jk} \in \text{GL}(n_k, \mathbb{C}), \quad j \in \{1, 2\}, \tag{4.42}$$

let boundary conditions (1.4) be of the form  $y(0) = Ay(1)$ , and let

$$Q(1-x) = A^{-1}Q(x)A, \quad x \in [0, \varepsilon], \quad \text{for some } \varepsilon > 0. \tag{4.43}$$

Then the system of root functions of the operator  $L_{C,D}(Q)$  is incomplete in  $L^2([0, 1]; \mathbb{C}^n)$  and its defect is infinite.

*Proof.* Due to the block structure of the matrices  $\tilde{B}$ ,  $A_1$  and  $A_2$ , one has  $AB + BA = 0$ . Since  $A_{jk}$  is nonsingular,  $\det A \neq 0$ . Therefore, Proposition 4.13 completes the result.  $\square$

**Remark 4.15.** *Note that in the case of  $2 \times 2$  Dirac system ( $B = \text{diag}(-1, 1)$ ,  $q_{11} \equiv q_{22} \equiv 0$ ) Proposition 4.13 turns into [45, Proposition 5.12]. Indeed, consider  $2 \times 2$  Dirac equation subject to the boundary conditions  $y_1(0) = \alpha_1 y_2(1)$ ,  $y_2(0) = \alpha_2 y_1(1)$ . Setting  $A = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}$ , one rewrites*

these conditions as  $y(0) = Ay(1)$ . Moreover, condition (4.43) turns into  $\alpha_1 q_{21}(1-x) = \alpha_2 q_{12}(x)$ ,  $x \in [0, \varepsilon] \cap [1-\varepsilon, 1]$ , for some  $\varepsilon > 0$ , i.e. coincides with the respective condition from [45]. Similar result for Sturm-Liouville operator subject to degenerate boundary conditions was proved earlier in [42].

## 5. THE RIESZ BASIS PROPERTY FOR ROOT FUNCTIONS

Here we investigate the Riesz basis property for operator  $L_{C,D}(Q)$  by reduction it to the operator  $L_{\tilde{C},\tilde{D}}(\tilde{Q})$  being a perturbation of a normal operator. To this end we find conditions for matrices  $C$  and  $D$  guarantying that  $L_{C,D}(0)$  is normal.

**Lemma 5.1.** (i) An operator  $L := L_{C,D}(0)$  is normal if and only if

$$CBC^* = DBD^*. \quad (5.1)$$

(ii) Boundary conditions (1.4) are regular, i.e.  $\det T_{izB}(C, D) \neq 0$  for each admissible  $z$ , whenever (5.1) is fulfilled.

(iii) If  $Q \in L^1([0, 1]; \mathbb{C}^{n \times n})$  and condition (5.1) is satisfied, then the system of root functions of the operator  $L_{C,D}(Q)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .

*Proof.* (i) It is easily seen that

$$LL^*y = L^*Ly = -(BB^*)^{-1}y'', \quad y \in W^{2,2}([0, 1]; \mathbb{C}^n). \quad (5.2)$$

Therefore,  $L$  is normal if and only if  $\text{dom}(L) = \text{dom}(L^*)$ , which is equivalent to

$$(Lf, g) = (f, L^*g), \quad f, g \in \text{dom}(L). \quad (5.3)$$

In turn, integrating by parts one gets that this identity is equivalent to

$$\langle B^{-1}f(0), g(0) \rangle = \langle B^{-1}f(1), g(1) \rangle, \quad f, g \in \text{dom}(L). \quad (5.4)$$

Put  $\tilde{B} := \text{diag}(B^{-1}, -B^{-1})$  and equip the space  $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^n$  with the bilinear form

$$w(u, v) := \langle \tilde{B}u, v \rangle = \langle B^{-1}u_1, v_1 \rangle - \langle B^{-1}u_2, v_2 \rangle, \quad u = \text{col}(u_1, u_2), \quad v = \text{col}(v_1, v_2). \quad (5.5)$$

Now condition (5.4) takes the form

$$w(u, v) = 0, \quad u, v \in \mathcal{H}_1 := \ker \begin{pmatrix} C & D \end{pmatrix} := \{\text{col}(u_1, u_2) : Cu_1 + Du_2 = 0\}. \quad (5.6)$$

On the other hand, the equality  $CBC^* = DBD^*$  can be rewritten as

$$\langle B^{-1}BC^*h, BC^*k \rangle = \langle B^{-1}(-BD^*h), -BD^*k \rangle, \quad h, k \in \mathbb{C}^n. \quad (5.7)$$

Using (5.5) one rewrites this equality in the form

$$w(u, v) = 0, \quad u, v \in \mathcal{H}_2 := \{\text{col}(BC^*h, -BD^*h) : h \in \mathbb{C}^n\}. \quad (5.8)$$

Thus, to prove the statement it suffices to show that (5.6) is equivalent to (5.8). To this end we prove that  $\mathcal{H}_1$  is the right  $w$ -orthogonal complement of  $\mathcal{H}_2$ ,

$$\mathcal{H}_1 = \mathcal{H}_2^{[1]} := \{u \in \mathcal{H} : w(v, u) = 0, v \in \mathcal{H}_2\}. \quad (5.9)$$

Indeed, if

$$v = \text{col}(BC^*h, -BD^*h) \in \mathcal{H}_2 \quad \text{and} \quad u = \text{col}(u_1, u_2) \in \mathcal{H}, \quad (5.10)$$

then

$$w(v, u) = \langle B^{-1}(BC^*)h, u_1 \rangle - \langle B^{-1}(-BD^*)h, u_2 \rangle = \langle h, Cu_1 + Du_2 \rangle, \quad h \in \mathbb{C}^n. \quad (5.11)$$

It follows that  $w(v, u) = 0$  for each  $v \in \mathcal{H}_2$  if and only if  $Cu_1 + Du_2 = 0$ , i.e.  $u \in \mathcal{H}_1$ . Next, maximality condition (1.6) yields  $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = n$ .



Now, if (5.8) is satisfied, then  $\mathcal{H}_2 \subset \mathcal{H}_2^{[\perp]} = \mathcal{H}_1$ . Since  $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$ , one has  $\mathcal{H}_1 = \mathcal{H}_2$  and (5.6) is fulfilled. The opposite implication is derived similarly.

(ii) Since  $L = L_{C,D}(0)$  is normal, condition (5.6) is satisfied. Let

$$\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_{2n}^{-1} \quad (5.12)$$

be the eigenvalues of  $\tilde{B}$  and let  $e_1, e_2, \dots, e_{2n}$  be the corresponding normalized eigenvectors. Note that

$$\beta_k = -\beta_{n+k} = b_k, \quad k \in \{1, \dots, n\}. \quad (5.13)$$

For every admissible  $z$ , i.e. for  $z$  satisfying

$$\operatorname{Re}(izb_k) \neq 0, \quad k \in \{1, \dots, n\}, \quad (5.14)$$

we put

$$\mathcal{H}_z := \operatorname{span}\{e_k : \operatorname{Re}(iz\beta_k) > 0\}. \quad (5.15)$$

Since

$$\beta_{n+k} = -\beta_k, \quad k \in \{1, \dots, n\}, \quad (5.16)$$

then  $\dim \mathcal{H}_z = n$  for every admissible  $z$ . Next we note that

$$T_{izB}(C, D) = (C \ D)|_{\mathcal{H}_z}. \quad (5.17)$$

Therefore,

$$\det T_{izB}(C, D) \neq 0 \quad \Leftrightarrow \quad \ker (C \ D) \cap \mathcal{H}_z = \{0\}. \quad (5.18)$$

Let  $u \in \mathcal{H}_z$ . Then

$$u = \sum_{\operatorname{Re}(iz\beta_k) > 0} c_k e_k, \quad (5.19)$$

for some  $c_1, \dots, c_n \in \mathbb{C}$ , and

$$\operatorname{Re}(iz\langle u, \tilde{B}u \rangle) = \sum_{\operatorname{Re}(iz\beta_k) > 0} |c_k|^2 \operatorname{Re}(iz\overline{\beta_k^{-1}}) = \sum_{\operatorname{Re}(iz\beta_k) > 0} \frac{|c_k|^2}{|\beta_k|^2} \operatorname{Re}(iz\beta_k). \quad (5.20)$$

Hence

$$\operatorname{Re}(iz\langle u, \tilde{B}u \rangle) > 0, \quad u \in \mathcal{H}_z \setminus \{0\}. \quad (5.21)$$

On the other hand, due to (5.6),

$$\langle u, \tilde{B}u \rangle = \overline{\langle \tilde{B}u, u \rangle} = 0, \quad u \in \ker (C \ D). \quad (5.22)$$

Combining this fact with (5.21) one obtains  $\ker (C \ D) \cap \mathcal{H}_z = \{0\}$  and we are done.

(iii) It follows from (ii) that boundary conditions (1.4) are weakly  $B$ -regular. Now the completeness and minimality of the root functions of the operator  $L_{C,D}(Q)$  is implied by [45, Theorem 1.2].  $\square$

**Remark 5.2.** Let  $Q \in L^2([0, 1]; \mathbb{C}^{n \times n})$ . Then (unbounded) multiplication operator

$$Q : f \rightarrow Q(x)f, \quad f \in L^2([0, 1]; \mathbb{C}^n), \quad (5.23)$$

is relatively compact with respect to  $L_{C,D}(0)$ . Therefore statement (iii) is implied by the classical Keldysh theorem (cf. [48, Theorem 4.3]) if in addition the spectrum of  $L_{C,D}(0)$  lies on the union of rays

$$\{\lambda \in \mathbb{C} : \arg \lambda = \varphi_k\}, \quad k \in \{1, \dots, n\}. \quad (5.24)$$

Recall the following definitions from [27] and [48].

**Definition 5.3.** (i) A sequence  $\{f_k\}_{k=1}^\infty$  of vectors in  $\mathfrak{H}$  is called a **Riesz basis** if it admits a representation  $f_k = Te_k$ ,  $k \in \mathbb{N}$ , where  $\{e_k\}_{k=1}^\infty$  is an orthonormal basis in  $\mathfrak{H}$  and  $T : \mathfrak{H} \rightarrow \mathfrak{H}$  is a bounded operator with bounded inverse.

(ii) A sequence of subspaces  $\{\mathfrak{H}_k\}_{k=1}^\infty$  is called a **Riesz basis of subspaces** in  $\mathfrak{H}$  if there exists a complete sequence of mutually orthogonal subspaces  $\{\mathfrak{H}'_k\}_{k=1}^\infty$  and a bounded operator  $T$  in  $\mathfrak{H}$  with bounded inverse such that  $\mathfrak{H}_k = T\mathfrak{H}'_k$ ,  $k \in \mathbb{N}$ .

(iii) A sequence  $\{f_k\}_{k=1}^\infty$  of vectors in  $\mathfrak{H}$  is called a **Riesz basis with parentheses** if each its finite subsequence is linearly independent, and there exists an increasing sequence  $\{n_k\}_{k=0}^\infty \subset \mathbb{N}$  such that  $n_0 = 1$  and the sequence  $\mathfrak{H}_k := \text{span}\{f_j\}_{j=n_{k-1}}^{n_k-1}$ , forms a Riesz basis of subspaces in  $\mathfrak{H}$ . Subspaces  $\mathfrak{H}_k$  are called blocks.

To state the next result we need the following definition.

**Definition 5.4.** Let  $\{\varphi_k\}_{k=1}^n$  be a sequence of angles,  $\varphi_k \in (-\pi, \pi]$ , and  $\varepsilon > 0$ . Numbers  $\lambda, \mu \in \mathbb{C}$  are called  $\varepsilon$ -close with respect to  $\{\varphi_k\}_{k=1}^n$ , if for some  $k \in \{1, \dots, n\}$  we have

$$\lambda, \mu \in \{z \in \mathbb{C} : |\arg z - \varphi_k| < \varepsilon\} \quad \text{and} \quad |\text{Re}(e^{-i\varphi_k}(\lambda - \mu))| < \varepsilon. \quad (5.25)$$

In other words,  $\lambda$  and  $\mu$  are  $\varepsilon$ -close if for some  $k$  they belong to a small angle with the bisectrix

$$l_+(\varphi_k) := \{\lambda \in \mathbb{C} : \arg \lambda = \varphi_k\} \quad (5.26)$$

and their projections on this ray are close.

Let  $A$  be an operator with compact resolvent and let  $\Omega$  be a bounded subset of  $\mathbb{C}$ . We put

$$N(\Omega, A) := \sum_{\lambda \in \sigma(A) \cap \Omega} m_a(\lambda, A) = \sum_{\lambda \in \sigma(A) \cap \Omega} \dim \mathcal{R}_\lambda(A). \quad (5.27)$$

Our investigation of the Riesz basis property of the operator  $L_{C,D}$  is based on the following statement that can easily be extracted from [49] and [48, §I.6].

**Proposition 5.5.** Let  $\mathfrak{H}$  be a separable Hilbert space and let  $G$  be a normal operator with compact resolvent in  $\mathfrak{H}$ . Assume that the spectrum of  $G$  lies on the union of rays  $l_+(\varphi_1), \dots, l_+(\varphi_n)$ , and

$$\sup_{z \in \mathbb{C}} N(\mathbb{D}(z), G) < \infty, \quad \mathbb{D}(z) := \{\zeta \in \mathbb{C} : |\zeta - z| < 1\}. \quad (5.28)$$

Finally, let  $T$  be a bounded operator in  $\mathfrak{H}$  and let  $\varepsilon > 0$  be arbitrarily small. Then the system of root vectors of the operator  $A = G + T$  forms a Riesz basis with parentheses in  $\mathfrak{H}$ , where each block is constituted by the root subspaces corresponding to the eigenvalues of  $A$  that are mutually  $\varepsilon$ -close with respect to the sequence  $\{\varphi_k\}_{k=1}^n$ .

*Proof.* Since  $T$  is bounded, it is relatively compact with respect to  $G$ . Hence by [48, Corollary 3.7], all but finitely many eigenvalues of  $A = G + T$  belong to the union of non-overlapping sectors

$$\Omega_j(\varepsilon) := \{\lambda \in \mathbb{C} : |\arg \lambda - \varphi_j| < \varepsilon\}, \quad j \in \{1, \dots, n\}. \quad (5.29)$$

Fix  $j \in \{1, \dots, n\}$  and set  $G_j := e^{-i\varphi_j}G$ . Condition (5.28) implies condition (6.21) of [48, Lemma 6.8],

$$\sup_{k \in \mathbb{N}} N((r_k - qr_k^p, r_k + qr_k^p), G_j) < \infty, \quad (5.30)$$

with  $p = 0$ , any  $q > 0$  and any increasing sequence  $\{r_k\}_{k=1}^\infty$ . Let  $\{\lambda_{j,k}\}_{k=1}^\infty$  be the sequence of eigenvalues of  $A$  belonging to  $\Omega_j(\varepsilon)$  and ordered in ascending order of  $\text{Re}(e^{-i\varphi_j}\lambda_{j,k})$ . Put

$$r_k := \text{Re}(e^{-i\varphi_j}\lambda_{j,k}) - \varepsilon/2, \quad k \in \mathbb{N}. \quad (5.31)$$

Applying [48, Lemma 6.8] to the operator  $G_j$  with  $p = 0$ ,  $q = \|T\| + 4\varepsilon$  and the above sequence  $\{r_k\}_{k=1}^\infty$ , we conclude that there exists

$$x_k \in (r_k - \varepsilon/2, r_k + \varepsilon/2), \quad k \in \mathbb{N}, \quad (5.32)$$

such that the sequence  $\{x_k\}_{k=1}^\infty$  is strictly monotone and the sequence of subspaces

$$\mathfrak{H}_{j,k} := \text{span}\{\mathcal{R}_{\lambda_{j,s}}(A) : x_k \leq \text{Re}(e^{-i\varphi_j} \lambda_{j,s}) < x_{k+1}\}, \quad k \in \mathbb{N}, \quad (5.33)$$

forms a Riesz basis of subspaces in its closed linear span. It follows from definition of  $r_k$  and  $x_k$  that

$$\text{Re}(e^{-i\varphi_j} \lambda_{j,k}) - \varepsilon < x_k < \text{Re}(e^{-i\varphi_j} \lambda_{j,k}), \quad k \in \mathbb{N}. \quad (5.34)$$

Hence root subspaces of  $A$  corresponding to the eigenvalues of  $A$ , that are not  $\varepsilon$ -close with respect to  $\{\varphi_k\}_{k=1}^n$ , belong to different blocks. Let  $\lambda'_1, \dots, \lambda'_m$  be the sequence of eigenvalues of  $A$  not belonging to the union of sectors  $\cup_{j=1}^n \Omega_j(\varepsilon)$ . Clearly, the family of subspaces

$$\{\mathcal{R}_{\lambda'_k}\}_{k=1}^m, \{\mathfrak{H}_{1,k}\}_{k=1}^\infty, \dots, \{\mathfrak{H}_{n,k}\}_{k=1}^\infty, \quad (5.35)$$

forms a Riesz basis of subspaces in its closed linear span. Since the latter spans the system of root vectors of the operator  $A$ , the Keldysh theorem (cf. [48, Theorem 4.3]) yields its completeness in  $\mathfrak{H}$ . Therefore, the system of root vectors of the operator  $A$  forms a Riesz basis with parentheses having the required properties of the blocks.  $\square$

Now we are ready to prove our main result on the Riesz basis property of BVP (1.2)–(1.4).

**Theorem 5.6.** *Let  $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$  and*

$$B = \text{diag}(B_j)_{j=1}^r, \quad C = \text{diag}(C_j)_{j=1}^r, \quad D = \text{diag}(D_j)_{j=1}^r, \quad (5.36)$$

where

$$B_j = \begin{pmatrix} b_{j1} I_{n_j} & 0 \\ 0 & b_{j2} I_{n_j} \end{pmatrix}, \quad b_{j1} b_{j2}^{-1} \in (-\infty, 0), \quad (5.37)$$

$$C_j = \begin{pmatrix} C_{j1} & C_{j2} \\ 0 & 0 \end{pmatrix}, \quad D_j = \begin{pmatrix} 0 & 0 \\ D_{j1} & D_{j2} \end{pmatrix}, \quad C_{j1}, C_{j2}, D_{j1}, D_{j2} \in \text{GL}(n_j, \mathbb{C}). \quad (5.38)$$

Then the system of root functions of the operator  $A := L_{C,D}(Q)$  forms a Riesz basis with parentheses in  $L^2([0, 1]; \mathbb{C}^n)$ , where each block is constituted by the root subspaces corresponding to the eigenvalues of  $A$  that are mutually  $\varepsilon$ -close with respect to the sequence of angles

$$\{-\varphi_1, \dots, -\varphi_r, \pi - \varphi_1, \dots, \pi - \varphi_r\}. \quad (5.39)$$

Here  $\varphi_j = \arg(b_{j1} - b_{j2})$ ,  $j \in \{1, \dots, r\}$ , and  $\varepsilon > 0$  is sufficiently small.

*Proof.* First we show that the operator  $L_{C,D}(Q)$  is similar to the operator  $L_{\tilde{C}, \tilde{D}}(\tilde{Q})$  with the same matrix  $B$  and matrices  $\tilde{C}, \tilde{D}$  satisfying (5.1). To this end we use the gauge transform  $W : y \rightarrow W(x)y$ , with  $W(\cdot)$  satisfying

$$W(x)B = BW(x), \quad x \in [0, 1], \quad (5.40)$$

$$W(\cdot) \in C^1([0, 1]; \mathbb{C}^{n \times n}), \quad W^{-1}(\cdot) \in C([0, 1]; \mathbb{C}^{n \times n}). \quad (5.41)$$

Then the operator  $L_{C,D}(Q)$  is transformed into the operator  $L_{\tilde{C}, \tilde{D}}(\tilde{Q}) = W^{-1}L_{C,D}(Q)W$  with the same  $B$ , and matrices  $\tilde{C}, \tilde{D}, \tilde{Q}(\cdot)$  given by

$$\tilde{C} := CW(0), \quad \tilde{D} := DW(1), \quad \tilde{Q}(x) := W^{-1}(x)Q(x)W(x) - iW^{-1}(x)B^{-1}W'(x). \quad (5.42)$$

Since  $W(\cdot), W'(\cdot), W^{-1}(\cdot), Q(\cdot) \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$ , then  $\tilde{Q} \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$ .

Due to the block diagonal structure (5.36)–(5.38) of the matrices  $B$ ,  $C_j$ , and  $D_j$ , we can choose  $W_0, W_1 \in \text{GL}(n, \mathbb{C})$  such that  $W_k B = B W_k$ ,  $k \in \{0, 1\}$ , and

$$C W_0 = \text{diag}(\tilde{C}_j)_{j=1}^r, \quad \tilde{C}_j := \begin{pmatrix} I_{n_j} & b_j I_{n_j} \\ 0 & 0 \end{pmatrix}, \quad b_j := \left(-b_{j1} b_{j2}^{-1}\right)^{1/2}, \quad (5.43)$$

$$D W_1 = \text{diag}(\tilde{D}_j)_{j=1}^r, \quad \tilde{D}_j := \begin{pmatrix} 0 & 0 \\ I_{n_j} & b_j I_{n_j} \end{pmatrix}, \quad j \in \{1, \dots, r\}. \quad (5.44)$$

Choose any branch of logarithm and put  $\tilde{W} := \log(W_0^{-1} W_1)$ . Clearly,  $\tilde{W}$  is well defined since the matrix  $W_0^{-1} W_1$  is nonsingular. Hence  $W(x) := W_0 e^{x \tilde{W}}$  satisfies (5.40), (5.41) and  $W(0) = W_0$ ,  $W(1) = W_1$ . Define a gauge transform  $W : y \rightarrow W(x)y$ . In view of (5.42), (5.43), (5.44) the matrices  $\tilde{C}, \tilde{D}$  of the new operator  $L_{\tilde{C}, \tilde{D}}(\tilde{Q}) = W^{-1} L_{C,D}(Q) W$  are  $\tilde{C} = \text{diag}(\tilde{C}_j)_{j=1}^r$  and  $\tilde{D} = \text{diag}(\tilde{D}_j)_{j=1}^r$  where  $\tilde{C}_j$  and  $\tilde{D}_j$  are given by (5.43) and (5.44), respectively.

Straightforward calculation shows that

$$\tilde{C}_j B_j \tilde{C}_j^* = \tilde{D}_j B_j \tilde{D}_j^* = 0, \quad j \in \{1, \dots, r\}. \quad (5.45)$$

Hence  $\tilde{C} B \tilde{C}^* = \tilde{D} B \tilde{D}^* = 0$ . By Lemma 5.1, the operator  $G := L_{\tilde{C}, \tilde{D}}(0)$  is normal. Its spectrum coincides with the set of zeros of the characteristic determinant  $\Delta(\cdot) = \det(\tilde{C} + \tilde{D} \tilde{\Phi}(1, \cdot))$ . The fundamental matrix  $\tilde{\Phi}(\cdot, \lambda)$  of the operator  $L_{\tilde{C}, \tilde{D}}(0)$  is  $\tilde{\Phi}(x, \lambda) = e^{i B \lambda x}$ . Hence, in view of the block-diagonal structure of the matrices  $B, \tilde{C}, \tilde{D}$ , we obtain

$$\Delta(\lambda) = \prod_{j=1}^r \det(\tilde{C}_j + \tilde{D}_j e^{i B_j \lambda}) = \prod_{j=1}^r \det \begin{pmatrix} I_{n_j} & b_j I_{n_j} \\ e^{i b_{j1} \lambda} I_{n_j} & b_j e^{i b_{j2} \lambda} I_{n_j} \end{pmatrix} = \prod_{j=1}^r \left( b_j^{n_j} \cdot (e^{i b_{j2} \lambda} - e^{i b_{j1} \lambda})^{n_j} \right). \quad (5.46)$$

Hence

$$\sigma(G) = \left\{ \frac{2\pi k}{b_{j1} - b_{j2}} : k \in \mathbb{Z}, j \in \{1, \dots, r\} \right\}. \quad (5.47)$$

Thus  $\sigma(G)$  lies on the union of rays  $\{l_+(-\varphi_j)\}_1^r$  and  $\{l_+(\pi - \varphi_j)\}_1^r$ , where

$$\varphi_j = \arg(b_{j1} - b_{j2}), \quad j \in \{1, \dots, r\}. \quad (5.48)$$

Moreover,  $\sigma(G)$  is the union of a finite number of arithmetic progressions and multiplicities of eigenvalues are bounded, hence condition (5.28) is satisfied. Since  $\tilde{Q}(\cdot)$  is bounded, then, by Proposition 5.5, the system of root functions of the operator

$$\tilde{A} := L_{\tilde{C}, \tilde{D}}(\tilde{Q}) = L_{\tilde{C}, \tilde{D}}(0) + \tilde{Q} = G + \tilde{Q} \quad (5.49)$$

forms a Riesz basis with parentheses in  $\mathfrak{H}$ , where each block is constituted by the root subspaces corresponding to the mutually close eigenvalues of  $A$  in the sense of Definition 5.4. Since  $A = L_{C,D}(Q)$  is similar to  $\tilde{A}$ , the same is true for the root functions of the operator  $L_{C,D}(Q)$ .  $\square$

As a consequence of this result we obtain *the Riesz basis property of the system of root functions for Dirac system with general splitting boundary conditions*.

**Corollary 5.7.** *Let  $n = 2m$ ,  $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$  and let*

$$B = \text{diag}(b_1 I_m, b_2 I_m), \quad b_1 < 0 < b_2, \quad (5.50)$$

$$C = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ D_1 & D_2 \end{pmatrix}, \quad C_1, C_2, D_1, D_2 \in \text{GL}(m, \mathbb{C}). \quad (5.51)$$

Then the system of root functions of the operator  $L_{C,D}(Q)$  forms a Riesz basis with parentheses in  $L^2([0, 1]; \mathbb{C}^{n \times n})$ .

Similarly to Theorem 5.6 we can obtain the following result.

**Proposition 5.8.** *Let*

$$B = \text{diag}(b_1 I_{n_1}, \dots, b_r I_{n_r}), \quad n = n_1 + \dots + n_r, \quad (5.52)$$

$$C = \text{diag}(C_j)_{j=1}^r, \quad D = \text{diag}(D_j)_{j=1}^r, \quad C_j, D_j \in GL(n_j, \mathbb{C}), \quad j \in \{1, \dots, r\}, \quad (5.53)$$

and  $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$ . Then the system of root functions of the operator  $A := L_{C,D}(Q)$  forms a Riesz basis with parentheses in  $L^2([0, 1]; \mathbb{C}^n)$ , where each block is constituted by the root subspaces corresponding to the eigenvalues of  $A$  that are mutually  $\varepsilon$ -close with respect to the sequence of angles  $\{-\varphi_1, \dots, -\varphi_r, \pi - \varphi_1, \dots, \pi - \varphi_r\}$ . Here  $\varphi_j = \arg b_j$ ,  $j \in \{1, \dots, r\}$ , and  $\varepsilon > 0$  is sufficiently small.

*Proof.* The proof is similar to that of Theorem 5.6. At first choosing an appropriate gauge transform, we transform the operator  $L_{C,D}(Q)$  into  $L_{\tilde{C},\tilde{D}}(\tilde{Q})$  with  $\tilde{C}_j = \tilde{D}_j = I_{n_j}$ . It follows that the operator  $G := L_{\tilde{C},\tilde{D}}(0)$  is normal and its spectrum is of the form

$$\sigma(G) = \{2\pi k/b_j : k \in \mathbb{Z}, j \in \{1, \dots, r\}\}. \quad (5.54)$$

Hence the same argument as in the proof of Theorem 5.6 yields the result.  $\square$

A direct consequence of this result is the *Riesz basis property of the periodic (reps. antiperiodic) BVP* with general matrix  $B$ .

**Corollary 5.9.** *Let  $B = \text{diag}(b_1, \dots, b_n) \in GL(n, \mathbb{C})$ ,  $C = \pm D = I_n$  and  $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$ . Then the system of root functions of the operator  $L_{C,D}(Q)$  forms a Riesz basis with parentheses in  $L^2([0, 1]; \mathbb{C}^{n \times n})$ .*

**Remark 5.10.** *In the case of Dirac type systems ( $B = B^*$ ) we can extend the statements of Theorem 5.6 and Proposition 5.8 to the case of  $Q \in L^2([0, 1]; \mathbb{C}^{n \times n})$ . Indeed, it suffices to apply Theorem 2 from the recent paper [60] instead of the quoted results from [49] and [48, §I.6]. Note however, that in [60, Theorem 2] only the basis property instead of the Riesz basis property was stated.*

**Remark 5.11.** *The Riesz basis property for  $2 \times 2$  Dirac equation subject to splitting boundary conditions has been investigated in numerous papers [69, 70, 29, 13, 14]. The most general statement covering Corollary 5.7 (for  $n = 1$ ) was obtained by Djakov and Mityagin [14] who relaxed the assumption on a potential matrix to  $Q \in L^2([0, 1]; \mathbb{C}^2)$ . Moreover, these authors proved in [14] the Riesz basis property for  $2 \times 2$  Dirac equation subject to **general strictly regular boundary conditions**.*

*For  $2 \times 2$  Dirac system Corollary 5.9 was proved in [13] under weaker assumption  $Q \in L^2([0, 1]; \mathbb{C}^2)$ . Moreover, these authors found out [18] a criterion for the system of root functions of the periodic boundary value problem for  $2 \times 2$  Dirac equation to contain a Riesz basis (without parentheses). Similar result for Sturm-Liouville operator  $-\frac{d^2}{dx^2} + q$  was obtained by different methods in [24, 25] and [18]. Both criteria are formulated directly in terms of periodic and Dirichlet eigenvalues. Moreover, in [16, Theorem 13], [18, Theorem 19] (see also [15]) it is established criteria for eigenfunctions and associated functions to form a Riesz basis for periodic 1D Dirac operator (resp. Sturm-Liouville operator) in terms of the Fourier coefficients of  $Q$  (resp.  $q$ ). Equivalence of this formulation to that in terms of periodic and Dirichlet eigenvalues*

is explained in [18, Theorem 24]. Let us mention in this connection the paper [61] where Riesz basis property for periodic Sturm-Liouville operator was obtained under certain explicit sufficient conditions in terms of Fourier coefficients of a potential  $q$ .

In the simplest case  $B = I_n$  we can indicate a criterion for the system of root functions of the operator  $L_{C,D}(Q)$  to form a Riesz basis with parentheses.

**Corollary 5.12.** *Let  $B = I_n$  and  $Q \in L^1([0, 1]; \mathbb{C}^{n \times n})$ . Then the system of root functions of the operator  $L_{C,D}(Q)$  forms a Riesz basis with parentheses in  $L_2([0, 1]; \mathbb{C}^n)$  if and only if  $\det(C \cdot D) \neq 0$ .*

*Proof.* Applying the gauge transform  $y \rightarrow W(x)y$  with  $W(\cdot)$  described in the beginning of the proof of Proposition 3.4, we see that the operator  $L_{C,D}(Q)$  is similar to the operator  $L_{\tilde{C},\tilde{D}}(0)$  with  $\tilde{C} = C$ ,  $\tilde{D} = DW(1)$  and zero potential matrix. Further, since  $B = I_n$ , then  $T_B(\tilde{C}, \tilde{D}) = DW(1)$  and  $T_{-B}(\tilde{C}, \tilde{D}) = C$ . Hence,  $\det(C \cdot D) \neq 0$  if and only if  $\det T_B(\tilde{C}, \tilde{D}) \cdot \det T_{-B}(\tilde{C}, \tilde{D}) \neq 0$ . Therefore, by [45, Proposition 4.6], the system of root functions of the operator  $L_{\tilde{C},\tilde{D}}(0)$  has infinite defect, whenever  $\det(C \cdot D) = 0$ . On the other hand, if  $\det(C \cdot D) \neq 0$  then, by Proposition 5.8, applied with  $r = 1$  and  $Q = 0$ , the system of root functions of the operator  $L_{\tilde{C},\tilde{D}}(0)$  forms a Riesz basis with parentheses. Similarity of the operators  $L_{C,D}(Q)$  and  $L_{\tilde{C},\tilde{D}}(0)$  completes the proof.  $\square$

## 6. GENERAL PROPERTIES OF THE RESOLVENT AND SPECTRAL SYNTHESIS

**6.1. General properties of the resolvent.** Let  $\mathfrak{H} := L^2([0, 1]; \mathbb{C}^n)$ . We start with the explicit form of the resolvent of the operator  $L_{C,D}(Q)$  in terms of the fundamental matrix solution  $\Phi(x, \lambda)$ . This statement is of folklore character (cf. [2, Theorem 9.4.1]). We present the proof for the sake of completeness.

**Lemma 6.1.** *Assume that BVP (1.2)–(1.4) is non-degenerate, i.e.  $\Delta(\cdot) \not\equiv 0$ . Then for any  $\lambda \in \mathbb{C}$  with  $\Delta(\lambda) \neq 0$  the Green function of the problem (1.2)–(1.4) is*

$$G(x, t; \lambda) = \begin{cases} \Phi(x, \lambda)(C + D\Phi(1, \lambda))^{-1}C\Phi^{-1}(t, \lambda)iB, & 0 \leq t \leq x, \\ -\Phi(x, \lambda)(C + D\Phi(1, \lambda))^{-1}D\Phi(1, \lambda)\Phi^{-1}(t, \lambda)iB, & x < t \leq 1. \end{cases} \quad (6.1)$$

Moreover,

$$G(x, x - 0; \lambda) - G(x, x + 0; \lambda) = iB, \quad x \in (0, 1). \quad (6.2)$$

*Proof.* Consider the non-homogenous system

$$-iB^{-1}y' + Q(x)y = \lambda y + f, \quad x \in [0, 1], \quad (6.3)$$

with  $f \in \mathfrak{H}$ . The general solution of this system is

$$y(x, \lambda) = \int_0^x \Phi(x, \lambda)\Phi^{-1}(t, \lambda)iBf(t)dt + \Phi(x, \lambda)y(0, \lambda), \quad (6.4)$$

where  $y(0, \lambda) = \text{col}(y_1(0, \lambda), \dots, y_n(0, \lambda)) \in \mathbb{C}^n$  is arbitrary vector. Inserting this expression into (1.4) we get after straightforward calculations

$$y(0, \lambda) = -(C + D\Phi(1, \lambda))^{-1}D\Phi(1, \lambda) \int_0^1 \Phi^{-1}(t, \lambda)iBf(t)dt =: K(\lambda)f. \quad (6.5)$$



Combining this expression with (6.4) we arrive at

$$\begin{aligned} y(x, \lambda) &= \int_0^x \Phi(x, \lambda) \left( I_n - (C + D\Phi(1, \lambda))^{-1} D\Phi(1, \lambda) \right) \Phi^{-1}(t, \lambda) iBf(t) dt \\ &\quad + \int_x^1 -\Phi(x, \lambda) (C + D\Phi(1, \lambda))^{-1} D\Phi(1, \lambda) \Phi^{-1}(t, \lambda) iBf(t) dt = \int_0^1 G(x, t; \lambda) f(t) dt, \end{aligned} \quad (6.6)$$

where  $G(x, t; \cdot)$  is given by (6.1). Formula (6.2) directly follows from (6.1).  $\square$

Combining (6.4) with (6.5) we get the following alternative representation for the resolvent.

**Corollary 6.2.** *Assume the conditions of Lemma 6.1, i.e.  $\rho(L_{C,D}(Q)) \neq \emptyset$ . Then for  $f \in \mathfrak{H}$  and  $\lambda \in \rho(L_{C,D}(Q))$*

$$(L_{C,D}(Q) - \lambda)^{-1} f = (L_{I_n,0}(Q) - \lambda)^{-1} f + \Phi(\cdot, \lambda) K(\lambda) f, \quad (6.7)$$

where  $K(\lambda) : \mathfrak{H} \rightarrow \mathbb{C}^n$  is given by (6.5) and  $(L_{I_n,0}(Q) - \lambda)^{-1}$  is Volterra operator of the form

$$((L_{I_n,0}(Q) - \lambda)^{-1} f)(x) = \int_0^x \Phi(x, \lambda) \Phi^{-1}(t, \lambda) iBf(t) dt. \quad (6.8)$$

**Theorem 6.3.** *For  $\lambda \in \rho(L_{C_1,D_1}(Q_1)) \cap \rho(L_{C_2,D_2}(Q_2))$  the following inclusion holds*

$$(L_{C_1,D_1}(Q_1) - \lambda)^{-1} - (L_{C_2,D_2}(Q_2) - \lambda)^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \quad (6.9)$$

Moreover, the following trace formula holds

$$\begin{aligned} &\text{tr}((L_{C_1,D_1}(Q_1) - \lambda)^{-1} - (L_{C_2,D_2}(Q_2) - \lambda)^{-1}) \\ &= \text{tr} \int_0^1 (\Phi_1(x, \lambda) (C_1 + D_1\Phi_1(1, \lambda))^{-1} C_1 \Phi_1^{-1}(x, \lambda) \\ &\quad - \Phi_2(x, \lambda) (C_2 + D_2\Phi_2(1, \lambda))^{-1} C_2 \Phi_2^{-1}(x, \lambda)) iB dx, \end{aligned} \quad (6.10)$$

where  $\Phi_j(\cdot, \lambda)$  is the fundamental matrix of the equation  $L(Q_j)y = \lambda y$  satisfying  $\Phi_j(0, \lambda) = I_n$ ,  $j \in \{1, 2\}$ .

*Proof.* (i) Put  $T_{11} := L_{C_1,D_1}(Q_1)$  and  $T_{22} := L_{C_2,D_2}(Q_2)$ . Consider the auxiliary operators  $T_j := L_{C,D}(Q_j)$ ,  $j \in \{1, 2\}$ , where  $C = I_n$  and  $D = 0$ . Clearly,  $T_j$  corresponds to the initial value problem

$$-iB^{-1}y' + Q_j(x)y = 0, \quad y(0) = 0. \quad (6.11)$$

Hence  $\rho(T_1) = \rho(T_2) = \mathbb{C}$ . Therefore, for  $\lambda \in \rho(T_{11}) \cap \rho(T_{22})$  we have

$$\begin{aligned} (T_{11} - \lambda)^{-1} - (T_{22} - \lambda)^{-1} &= ((T_{11} - \lambda)^{-1} - (T_1 - \lambda)^{-1}) \\ &\quad + ((T_1 - \lambda)^{-1} - (T_2 - \lambda)^{-1}) + ((T_2 - \lambda)^{-1} - (T_{22} - \lambda)^{-1}). \end{aligned} \quad (6.12)$$

It follows from (6.7) that the first and the third summands in (6.12) are operators of finite rank,

$$\dim \left( \text{ran}((T_{11} - \lambda)^{-1} - (T_1 - \lambda)^{-1}) \right) \leq n, \quad (6.13)$$

$$\dim \left( \text{ran}((T_2 - \lambda)^{-1} - (T_{22} - \lambda)^{-1}) \right) \leq n. \quad (6.14)$$

Next, according to [23, Theorem 2.7], for each  $x \in [0, 1]$  the matrix  $Q(x) := Q_2(x) - Q_1(x)$  admits the generalized polar decomposition

$$Q(x) = U(x) \cdot |Q(x)| = |Q^*(x)| \cdot U(x) = |Q^*(x)|^{1/2} \cdot U(x) \cdot |Q(x)|^{1/2}, \quad (6.15)$$

where  $|A| := (A^*A)^{1/2}$ ,  $A \in \mathbb{C}^{n \times n}$ , and  $U(x)$  is a unitary matrix,  $U^*(x) = U^{-1}(x)$ . Clearly,  $|Q(\cdot)|$  and  $|Q^*(\cdot)|$  are measurable matrix-function and  $U(\cdot)$  can be chosen to be measurable. In turn, these families induce a generalized polar decomposition of the (unbounded) multiplication operator  $Q : f(x) \rightarrow Q(x)f(x)$  in  $\mathfrak{H}$ , i.e.

$$Q = U \cdot |Q| = |Q^*| \cdot U = |Q^*|^{1/2} \cdot U \cdot |Q|^{1/2}, \quad |Q| = (Q^*Q)^{1/2}, \quad |Q^*| = (QQ^*)^{1/2}, \quad (6.16)$$

and  $|Q|$  denotes the multiplication operator in  $\mathfrak{H}$  with the matrix  $|Q(\cdot)|$ .

Let  $G_j(\cdot, \cdot; \lambda)$  be the Green function of the operator  $T_j$ ,  $j \in \{1, 2\}$ . It is easily seen that

$$(T_1 - \lambda)^{-1} - (T_2 - \lambda)^{-1} = \overline{(T_1 - \lambda)^{-1} |Q^*|^{1/2}} \cdot (U |Q|^{1/2} (T_2 - \lambda)^{-1}) = K_1 K_2, \quad (6.17)$$

where  $\overline{T}$  denotes the closure of the operator  $T$  and for  $f \in \mathfrak{H}$ ,

$$(K_1 f)(x) := \int_0^1 \left( G_1(x, t; \lambda) |Q^*(t)|^{1/2} \right) f(t) dt, \quad x \in [0, 1], \quad (6.18)$$

$$(K_2 f)(x) := \int_0^1 \left( U(x) |Q(x)|^{1/2} G_2(x, t; \lambda) \right) f(t) dt, \quad x \in [0, 1]. \quad (6.19)$$

It follows from (6.1) that the kernel

$$G_j(\cdot, \cdot; \lambda) \in L^\infty([0, 1] \times [0, 1]; \mathbb{C}^{n \times n}), \quad j \in \{1, 2\}. \quad (6.20)$$

Moreover, since

$$|Q(\cdot)|^{1/2}, |Q^*(\cdot)|^{1/2} \in L^2([0, 1]; \mathbb{C}^{n \times n}) \quad \text{and} \quad U(\cdot) \in L^\infty([0, 1]; \mathbb{C}^{n \times n}), \quad (6.21)$$

the operator  $K_j$  is of Hilbert-Schmidt class,  $K_j \in \mathfrak{S}_2(\mathfrak{H})$ ,  $j \in \{1, 2\}$ . Combining these relations with factorization identity (6.17) yields

$$(T_1 - \lambda)^{-1} - (T_2 - \lambda)^{-1} = K_1 K_2 \in \mathfrak{S}_1(\mathfrak{H}). \quad (6.22)$$

In turn, combining this relation with (6.12)–(6.14) we arrive at (6.9).

(ii) Let  $G_{jj}(\cdot, \cdot; \lambda)$  be the Green function of the operator  $T_{jj}$ ,  $j \in \{1, 2\}$ . By (i), the difference  $(T_{11} - \lambda)^{-1} - (T_{22} - \lambda)^{-1}$  is of trace class integral operator with the kernel

$$\tilde{G}(\cdot, \cdot; \lambda) := G_{11}(\cdot, \cdot; \lambda) - G_{22}(\cdot, \cdot; \lambda). \quad (6.23)$$

In view of (6.1) and (6.2) the kernel  $\tilde{G}(\cdot, \cdot; \lambda)$  is continuous,

$$\tilde{G}(\cdot, \cdot; \lambda) \in C([0, 1] \times [0, 1]; \mathbb{C}^{n \times n}). \quad (6.24)$$

On the other hand, if  $K$  is of trace class integral operator in  $L^2([0, 1]; \mathbb{C}^n)$  with continuous kernel  $K(\cdot, \cdot) \in \mathbb{C}^{n \times n}$ , then

$$\text{tr } K = \int_0^1 (\text{tr } K(x, x)) dx \quad (6.25)$$

(see [27, Corollary III.10.2]). Combining this result with formula (6.1) for the Green function  $G_{jj}(\cdot, \cdot; \lambda)$ ,  $j \in \{1, 2\}$ , yields (6.10).  $\square$

**Corollary 6.4.** *Let  $B = B^*$  and let  $C_*, D_* \in \mathbb{C}^{n \times n}$  be such that  $(L_{C,D}(Q))^* = L_{C_*,D_*}(Q^*)$  (see Lemma 2.1). Let also  $\Phi(\cdot, \lambda)$  and  $\Phi_*(\cdot, \lambda)$  be the fundamental matrices of equations  $L(Q)y = \lambda y$  and  $L(Q^*)y = \lambda y$ , respectively, satisfying  $\Phi(0, \lambda) = \Phi_*(\cdot, \lambda) = I_n$ . Finally, let  $\{\lambda_n\}_{n=1}^\infty$  be the sequence of all eigenvalues of  $L_{C,D}(Q)$ , counting multiplicity, and let the system of root functions*

of the operator  $L_{C,D}(Q)$  be complete in  $L^2([0, 1]; \mathbb{C}^n)$ . Then for  $\lambda \in \mathbb{R} \cap \rho(L_{C,D}(Q))$  the following identity holds

$$\sum_{n=1}^{\infty} \frac{-2\operatorname{Im} \lambda_n}{|\lambda_n - \lambda|^2} = \operatorname{tr} \int_0^1 (\Phi(x, \lambda)(C + D\Phi(1, \lambda))^{-1}C\Phi^{-1}(x, \lambda) - \Phi_*(x, \lambda)(C_* + D_*\Phi_*(1, \lambda))^{-1}C_*\Phi_*^{-1}(x, \lambda))Bdx. \quad (6.26)$$

*Proof.* By Theorem 6.3, the imaginary part of the resolvent  $(L_{C,D}(Q) - \lambda)^{-1}$  is of trace class operator,

$$(L_{C,D}(Q) - \lambda)^{-1} - (L_{C,D}(Q)^* - \lambda)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \lambda \in \mathbb{R} \cap \rho(L_{C,D}(Q)), \quad (6.27)$$

and trace formula (6.10) holds. Combining formula (6.10) with Livsic theorem (see [27, Theorem V.2.1]) yields the result.  $\square$

To state the next result we recall some properties of the classes  $\mathcal{S}_p(\mathfrak{H})$  and  $\mathcal{S}_p^0(\mathfrak{H})$ ,  $p \in (0, \infty)$ , introduced in the following definition.

**Definition 6.5.** Define for  $p > 0$

$$\mathcal{S}_p(\mathfrak{H}) = \{T \in \mathfrak{S}_\infty(\mathfrak{H}) \mid s_j(T) = O(j^{-1/p}) \text{ as } j \rightarrow \infty\}, \quad (6.28)$$

$$\mathcal{S}_p^0(\mathfrak{H}) = \{T \in \mathfrak{S}_\infty(\mathfrak{H}) \mid s_j(T) = o(j^{-1/p}) \text{ as } j \rightarrow \infty\}, \quad (6.29)$$

where  $s_j(T)$ ,  $j \in \mathbb{N}$ , denote the singular values ( $s$ -numbers) of  $T$  (i.e., the eigenvalues of  $(T^*T)^{1/2}$  ordered in decreasing magnitude, counting multiplicity).

Clearly  $\mathfrak{S}_p \subset \mathcal{S}_p^0 \subset \mathcal{S}_p$ . It is known that  $\mathcal{S}_p(\mathfrak{H})$  ( $\mathcal{S}_p^0(\mathfrak{H})$ ) is a two-sided (non-closed) ideal in  $\mathcal{B}(\mathfrak{H})$ . Clearly,  $\mathcal{S}_{p_1} \subset \mathcal{S}_{p_2}$  and  $\mathcal{S}_{p_1}^0 \subset \mathcal{S}_{p_2}^0$  if  $p_1 > p_2$ . The main property of the classes  $\mathcal{S}_p(\mathfrak{H})$  and  $\mathcal{S}_p^0(\mathfrak{H})$  we need in the sequel, is (see [27, §II.2])

$$\mathcal{S}_{p_1} \cdot \mathcal{S}_{p_2} \subset \mathcal{S}_p, \quad \text{and} \quad \mathcal{S}_{p_1} \cdot \mathcal{S}_{p_2}^0 \subset \mathcal{S}_p^0, \quad \text{where} \quad p^{-1} = p_1^{-1} + p_2^{-1}. \quad (6.30)$$

We need also a generalization of the known Ky-Fan lemma (see [27, Theorem II.2.3]).

**Lemma 6.6.** Let  $A, B \in \mathfrak{S}_\infty(\mathfrak{H})$ ,  $r > 0$ , and let the following conditions be satisfied

$$\lim_{k \rightarrow \infty} (k^r \cdot s_{n_k}(A)) = a, \quad \lim_{n \rightarrow \infty} (n^r \cdot s_n(B)) = 0, \quad (6.31)$$

where  $\{n_k\}_{k=1}^\infty$  is an increasing sequence of positive integers. Then

$$\lim_{k \rightarrow \infty} (k^r \cdot s_{n_k}(A + B)) = a. \quad (6.32)$$

*Proof.* We follow the proof of Ky-Fan lemma [27, Theorem II.2.3]. Fix  $\varepsilon \in (0, 1)$  and for any  $k \in \mathbb{N}$  define  $j = j_{\varepsilon, k} := k - \lfloor \varepsilon k \rfloor \in \mathbb{N}$ . Then the Ky-Fan inequality,

$$s_n(A + B) \leq s_m(A) + s_{n-m+1}(B), \quad n \geq m, \quad (6.33)$$

implies for each  $k \in \mathbb{N}$

$$\begin{aligned} k^r \cdot s_{n_k}(A + B) &\leq \left(\frac{k}{j}\right)^r \cdot (j^r \cdot s_{n_j}(A)) + \left(\frac{k}{n_k - n_j + 1}\right)^r \cdot ((n_k - n_j + 1)^r \cdot s_{n_k - n_j + 1}(B)) \\ &\leq \frac{1}{(1 - \varepsilon)^r} \cdot (j^r \cdot s_{n_j}(A)) + \frac{1}{\varepsilon^r} \cdot ((n_k - n_j + 1)^r \cdot s_{n_k - n_j + 1}(B)). \end{aligned} \quad (6.34)$$

Here we have used that

$$k - \varepsilon k \leq j < k - \varepsilon k + 1 \quad \text{and} \quad n_k - n_j \geq k - j. \quad (6.35)$$

Note that

$$j_{\varepsilon,k} \rightarrow \infty \quad \text{and} \quad n_k - n_j + 1 \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty. \quad (6.36)$$

Hence tending  $k$  to infinity in (6.34) and using (6.31) we derive

$$\overline{\lim}_{k \rightarrow \infty} (k^r \cdot s_{n_k}(A + B)) \leq \frac{a}{(1 - \varepsilon)^r}. \quad (6.37)$$

Tending  $\varepsilon$  to zero here we get

$$\overline{\lim}_{k \rightarrow \infty} (k^r \cdot s_{n_k}(A + B)) \leq a. \quad (6.38)$$

Using inequality

$$s_{n_j}(A) \leq s_{n_k}(A + B) + s_{n_j - n_k + 1}(B), \quad j = k + \lfloor \varepsilon k \rfloor, \quad (6.39)$$

we obtain in a similar way that

$$\underline{\lim}_{k \rightarrow \infty} (k^r \cdot s_{n_k}(A + B)) \geq a. \quad (6.40)$$

One completes the proof by combining this inequality with (6.38).  $\square$

Now we are ready to find the asymptotic behavior of the  $s$ -numbers of the resolvent operator  $(L_{C,D}(Q) - \lambda)^{-1}$  for each fixed  $\lambda \in \rho(L_{C,D}(Q))$ .

**Proposition 6.7.** *Let  $\rho(L_{C,D}(Q)) \neq \emptyset$ . Then for any  $\lambda \in \rho(L_{C,D}(Q))$  the following inclusion holds*

$$(L_{C,D}(Q) - \lambda)^{-1} \in \mathcal{S}_1(\mathfrak{H}) \setminus \mathfrak{S}_1(\mathfrak{H}). \quad (6.41)$$

Moreover, the sequence  $\{s_k\}_{k \in \mathbb{N}}$  of singular values of the operator  $(L_{C,D}(Q) - \lambda)^{-1}$  can be decomposed into the union of  $n$  disjoint non-increasing subsequences  $\{s_{j,k}\}_{k \in \mathbb{N}}$ ,  $j \in \{1, \dots, n\}$ , satisfying

$$s_{j,k} = \frac{|b_j| + o(1)}{\pi k} \quad \text{as} \quad k \rightarrow \infty. \quad (6.42)$$

*Proof.* Put  $L := L_{C,D}(Q)$ . Since  $\rho(L) \neq \emptyset$ , its spectrum is discrete. Alongside the operator  $L_{C,D}(Q)$  consider the auxiliary operator  $L_0 := L_{I_n, -I_n}(0)$  corresponding to the periodic boundary value problem

$$-iB^{-1}y' = 0, \quad y(0) = y(1). \quad (6.43)$$

Straightforward calculation shows that the spectrum  $\sigma(L_0)$  is decomposed into  $n$  series

$$\{2\pi k/b_j\}_{k \in \mathbb{Z}}, \quad j \in \{1, \dots, n\}. \quad (6.44)$$

It is easily seen that the operator  $L_0$  is normal. Hence the singular values of the operator  $(L_0 - \lambda)^{-1}$  coincide with the absolute values of its eigenvalues. Hence the sequence  $\{s_k^{(0)}\}_{k \in \mathbb{N}}$  of singular values of  $(L_0 - \lambda)^{-1}$  can be reordered and decomposed into the union of  $n$  disjoint subsequences  $\{\tilde{s}_{j,k}^{(0)}\}_{k \in \mathbb{Z}}$ ,  $j \in \{1, \dots, n\}$ , such that  $\tilde{s}_{j,k}^{(0)} = \left| \frac{2\pi k}{b_j} - \lambda \right|^{-1}$ . Reordering the sequence  $\{\tilde{s}_{j,k}^{(0)}\}_{k \in \mathbb{Z}}$  in decreasing order of magnitude we obtain the sequence  $\{s_{j,k}^{(0)}\}_{k \in \mathbb{N}}$  satisfying

$$s_{j,k}^{(0)} = \frac{|b_j| + o(1)}{\pi k} \quad \text{as} \quad k \rightarrow \infty. \quad (6.45)$$

Therefore,  $s_k^{(0)} = O(k^{-1})$  as  $k \rightarrow \infty$  and hence

$$(L_0 - \lambda)^{-1} \in \mathcal{S}_1(\mathfrak{H}), \quad \lambda \in \rho(L_0). \quad (6.46)$$

Since  $\sigma(L_0)$  and  $\sigma(L)$  are at most countable,  $\rho(L_0) \cap \rho(L) \neq \emptyset$ . By Theorem 6.3,

$$(L - \lambda_0)^{-1} - (L_0 - \lambda_0)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \lambda_0 \in \rho(L_0) \cap \rho(L). \quad (6.47)$$

Combining this relation with just established inclusion  $(L_0 - \lambda_0)^{-1} \in \mathcal{S}_1(\mathfrak{H})$  yields

$$(L - \lambda_0)^{-1} = (L_0 - \lambda_0)^{-1} + ((L - \lambda_0)^{-1} - (L_0 - \lambda_0)^{-1}) \in \mathcal{S}_1(\mathfrak{H}). \quad (6.48)$$

As an immediate consequence of (6.47) one gets that the sequence  $\{s_k^{(1)}\}_{k \in \mathbb{N}}$  of  $s$ -numbers of the operator  $(L_0 - \lambda_0)^{-1} - (L - \lambda_0)^{-1}$  satisfies

$$s_k^{(1)} = o(k^{-1}) \quad \text{as } k \rightarrow \infty. \quad (6.49)$$

Combining (6.45) with (6.49) and applying Lemma 6.6 we arrive at the desired asymptotic formula (6.42) for  $s$ -numbers of the operator  $(L - \lambda_0)^{-1}$ .

Next, noting that  $\mathcal{S}_1(\mathfrak{H})$  is two-sided ideal in  $\mathfrak{H}$  and using the Hilbert identity for the resolvent,

$$(L - \lambda)^{-1} = (L - \lambda_0)^{-1} + (\lambda_0 - \lambda)(L - \lambda)^{-1}(L - \lambda_0)^{-1}, \quad (6.50)$$

one gets  $(L - \lambda)^{-1} \in \mathcal{S}_1(\mathfrak{H})$  for  $\lambda \in \rho(L)$ . Moreover, since  $(L - \lambda)^{-1}, (L - \lambda_0)^{-1} \in \mathcal{S}_1(\mathfrak{H})$ , we obtain from (6.30) that

$$(L - \lambda)^{-1} \cdot (L - \lambda_0)^{-1} \in \mathcal{S}_1(\mathfrak{H}) \cdot \mathcal{S}_1(\mathfrak{H}) \subset \mathcal{S}_{1/2}(\mathfrak{H}) \subset \mathcal{S}_1^0(\mathfrak{H}). \quad (6.51)$$

Combining this relation with Lemma 6.6 yields the desired asymptotic formula for the  $s$ -numbers of the operator  $(L - \lambda)^{-1}$ ,  $\lambda \in \rho(L)$ . This implies that  $(L - \lambda)^{-1} \notin \mathfrak{S}_1(\mathfrak{H})$ , which completes the proof.  $\square$

Next we improve Theorem 6.3 (see formula (6.9)) assuming that  $Q_2 - Q_1 \in L^2([0, 1]; \mathbb{C}^{n \times n})$ .

**Corollary 6.8.** *Let  $Q_2 - Q_1 \in L^2([0, 1]; \mathbb{C}^{n \times n})$ . Then for  $\lambda \in \rho(L_{C_1, D_1}(Q_1)) \cap \rho(L_{C_2, D_2}(Q_2))$  the following inclusion holds*

$$(L_{C_1, D_1}(Q_1) - \lambda)^{-1} - (L_{C_2, D_2}(Q_2) - \lambda)^{-1} \in \mathcal{S}_{2/3}^0(\mathfrak{H}). \quad (6.52)$$

*Proof.* Following the proof of Theorem 6.3 it suffices to show that

$$(T_1 - \lambda)^{-1} - (T_2 - \lambda)^{-1} \in \mathcal{S}_{2/3}^0(\mathfrak{H}), \quad \lambda \in \mathbb{C}, \quad (6.53)$$

where  $T_j = L_{I_n, 0}(Q_j)$ ,  $j \in \{1, 2\}$ . Let  $G_2(\cdot, \cdot; \lambda)$  be the Green function of the operator  $T_2$ . By Lemma 6.1 (cf. formula (6.1)),

$$G_2(\cdot, \cdot; \lambda) \in L^\infty([0, 1] \times [0, 1]; \mathbb{C}^{n \times n}). \quad (6.54)$$

Combining this fact with the assumption

$$Q(\cdot) := Q_2(\cdot) - Q_1(\cdot) \in L^2([0, 1]; \mathbb{C}^{n \times n}), \quad (6.55)$$

one gets

$$Q(x) \cdot G_2(x, t; \lambda) \in L^2([0, 1] \times [0, 1]; \mathbb{C}^{n \times n}). \quad (6.56)$$

The latter means that  $Q(T_2 - \lambda)^{-1}$  is Hilbert-Schmidt operator,

$$Q(T_2 - \lambda)^{-1} \in \mathfrak{S}_2(\mathfrak{H}) \subset \mathcal{S}_2^0(\mathfrak{H}). \quad (6.57)$$

By Proposition 6.7,  $(T_1 - \lambda)^{-1} \in \mathcal{S}_1(\mathfrak{H})$ . Combining this fact with property (6.30) of classes  $\mathcal{S}_p(\mathfrak{H})$ , yields

$$(T_1 - \lambda)^{-1} - (T_2 - \lambda)^{-1} = (T_1 - \lambda)^{-1} \cdot (Q(T_2 - \lambda)^{-1}) \in \mathcal{S}_1(\mathfrak{H}) \cdot \mathcal{S}_2^0(\mathfrak{H}) \subset \mathcal{S}_{2/3}^0(\mathfrak{H}), \quad (6.58)$$

which completes the proof.  $\square$

**6.2. Spectral synthesis for dissipative Dirac type operators.** Recall that an operator  $T$  in a Hilbert space  $\mathfrak{H}$  is called accumulative (dissipative) if

$$\operatorname{Im}(Tf, f) \leq 0 \ (\geq 0), \quad f \in \operatorname{dom}(T). \quad (6.59)$$

Note that the accumulativity (dissipativity) of BVP (1.2)–(1.4) implies  $B = B^*$ . Therefore, to investigate accumulative (dissipative) BVP we are forced to consider Dirac type operators only. At first we express accumulativity (dissipativity) of the operator  $L_{C,D}(Q)$  in terms of matrices  $B, C, D$  and  $Q(\cdot)$ .

**Lemma 6.9.** *Let  $B = B^*$ . The operator  $L_{C,D}(Q)$  is accumulative (dissipative) if and only if  $\operatorname{Im} Q \leq 0$  ( $\operatorname{Im} Q \geq 0$ ) and*

$$CBC^* - DBD^* \geq 0 \ (\leq 0). \quad (6.60)$$

*In particular, the operator  $L_{C,D}(Q)$  is selfadjoint if and only if  $Q = Q^*$  and  $CBC^* = DBD^*$ .*

*Proof.* Integrating by parts and noting that  $B = B^*$  one easily gets for  $f \in \operatorname{dom}(L_{C,D}(Q))$ ,

$$2\operatorname{Im}(L_{C,D}(Q)f, f) = \langle B^{-1}f(0), f(0) \rangle - \langle B^{-1}f(1), f(1) \rangle + \int_0^1 \langle 2\operatorname{Im} Q(x)f(x), f(x) \rangle dx. \quad (6.61)$$

Let us show that  $L_{C,D}(Q)$  is accumulative if and only if  $\operatorname{Im} Q \leq 0$  and

$$\langle B^{-1}h_0, h_0 \rangle - \langle B^{-1}h_1, h_1 \rangle \leq 0 \quad \text{whenever} \quad Ch_0 + Dh_1 = 0, \quad h_0, h_1 \in \mathbb{C}^n. \quad (6.62)$$

Indeed, if  $\operatorname{Im} Q \leq 0$  then due to (6.61) the inequality  $\operatorname{Im}(L_{C,D}(Q)f, f) \leq 0$  is implied by (6.62). Conversely, let  $L_{C,D}(Q)$  be accumulative. Choose any  $f \in \operatorname{dom}(L_{C,D}(Q))$  with  $f(0) = f(1) = 0$  and substitute it in (6.61). Then the inequality  $\operatorname{Im}(L_{C,D}(Q)f, f) \leq 0$  turns into

$$\int_0^1 \langle \operatorname{Im} Q(x)f(x), f(x) \rangle dx \leq 0, \quad f \in \operatorname{dom}(L_{C,D}(Q)) \cap W_0^{1,1}([0, 1]; \mathbb{C}^n), \quad (6.63)$$

which yields  $\operatorname{Im} Q(x) \leq 0$ ,  $x \in [0, 1]$ . Further, to extract (6.62) from the inequality  $\operatorname{Im} L_{C,D}(Q) \leq 0$  we fix  $h_0, h_1 \in \mathbb{C}^n$  with  $Ch_0 + Dh_1 = 0$  and substitute in (6.61) function  $f \in \operatorname{dom}(L_{C,D}(Q))$  such that

$$f(0) = h_0, \quad f(1) = h_1 \quad \text{and} \quad \operatorname{supp} f \subset [0, \varepsilon] \cup [1 - \varepsilon, 1], \quad (6.64)$$

where  $\varepsilon > 0$  is sufficiently small. Thus, to prove the statement it suffices to show that inequality (6.62) is equivalent to (6.60).

As in the proof of Lemma 5.1, we put  $\tilde{B} := \operatorname{diag}(B^{-1}, -B^{-1})$  and consider  $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^n$  as the Pontryagin space equipped with the bilinear form  $w$  given by (5.5). Clearly, the inertia indices of  $\mathcal{H}$  are  $\kappa_{\pm} = \kappa_{\pm}(\tilde{B}) = n$ , where  $\kappa_+(A)$  ( $\kappa_-(A)$ ) denotes the number of positive (resp. negative) eigenvalues of a matrix  $A = A^*$ . Let  $\mathcal{H}_1 := \ker \begin{pmatrix} C & D \end{pmatrix} \subset \mathcal{H}$ . Then it is clear that (6.62) is satisfied if and only if the subspace  $\mathcal{H}_1$  is non-positive in  $\mathcal{H}$ , i.e.

$$\mathcal{H}_1 \subset \{u \in \mathcal{H} : \langle \tilde{B}u, u \rangle \leq 0\}. \quad (6.65)$$

Further, condition (6.60) rewritten as

$$\langle C^*h, BC^*h \rangle \geq \langle D^*h, BD^*h \rangle, \quad h \in \mathbb{C}^n, \quad (6.66)$$

is equivalent to

$$\langle \tilde{B}u, u \rangle \geq 0, \quad u \in \mathcal{H}_2 := \{\operatorname{col}(BC^*h, -BD^*h) : h \in \mathbb{C}^n\}, \quad (6.67)$$

meaning the non-negativity of the subspace  $\mathcal{H}_2$ . Note that maximality condition (1.6) yields  $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = n$ . As it is proved in Lemma 5.1,  $\mathcal{H}_1$  is  $w$ -orthogonal complement of the subspace  $\mathcal{H}_2$ . Since  $w$ -orthogonal complement of a maximal non-positive subspace is the

maximal non-negative and vice versa, and taking into account that the inertia indices of  $\mathcal{H}$  are  $\kappa_{\pm} = n = \dim \mathcal{H}_1 = \dim \mathcal{H}_2$ , one derives that  $\mathcal{H}_1$  is non-positive in  $\mathcal{H}$  if and only if  $\mathcal{H}_2$  is non-negative in  $\mathcal{H}$ .  $\square$

**Lemma 6.10.** *Let  $B = B^*$  and let the operator  $L_{C,D}(0)$  be accumulative (dissipative). Then*

$$\det T_{-B}(C, D) \neq 0 \quad (\det T_B(C, D) \neq 0). \quad (6.68)$$

*Proof.* Denote by  $P_+$  and  $P_-$  the spectral projections onto "positive" and "negative" parts of the spectrum of a selfadjoint matrix  $B = B^*$ , respectively. Then

$$T_- := T_{-B}(C, D) = CP_+ + DP_-. \quad (6.69)$$

By Lemma 6.9, it suffices to show that  $\det T_- \neq 0$  is implied by (6.60). Let  $h_0 \in \ker T_-^*$ . Since

$$T_-^* = P_+C^* + P_-D^* \quad \text{and} \quad P_+P_- = P_-P_+ = 0, \quad (6.70)$$

one gets

$$P_+C^*h_0 = P_-D^*h_0 = 0. \quad (6.71)$$

Setting  $B_{\pm} := \pm P_{\pm}B$ , and noting that  $B = B_+ - B_-$ , we rewrite inequality (6.60) in the following form

$$\|B_+^{1/2}P_+C^*h\|^2 + \|B_-^{1/2}P_-D^*h\|^2 \geq \|B_-^{1/2}P_-C^*h\|^2 + \|B_+^{1/2}P_+D^*h\|^2, \quad h \in \mathbb{C}^n. \quad (6.72)$$

Substituting in this inequality  $h_0$  in place of  $h$  and using (6.71) we get  $P_-C^*h_0 = P_+D^*h_0 = 0$ . Combining these relations with (6.71), yields  $C^*h_0 = D^*h_0 = 0$ . Hence the maximality condition  $\ker(CC^* + DD^*) = \{0\}$  implies  $h_0 = 0$ . Therefore,  $\ker T_-^* = \{0\}$ , which yields (6.68).  $\square$

Thus, in the case of dissipative (accumulative) boundary conditions their regularity (1.9) is reduced to the solo condition  $\det T_- \neq 0$  ( $\det T_+ \neq 0$ ).

Passing to the spectral synthesis we recall the following definition.

**Definition 6.11.** (i) *A compact operator  $T$  in a separable Hilbert space  $\mathfrak{H}$  is called complete if the system of its root vectors is complete in  $\mathfrak{H}$ .*

(ii) *A compact complete operator  $T$  in  $\mathfrak{H}$  admits the spectral synthesis if for any invariant subspace  $\mathfrak{H}_1$  of  $T$  the restriction  $T|_{\mathfrak{H}_1}$  is complete in  $\mathfrak{H}_1$ .*

(iii) *A closed operator  $T$  in  $\mathfrak{H}$  with  $\rho(T) \neq \emptyset$  is called complete if its resolvent is compact and complete. We say that  $T$  admits the spectral synthesis if its resolvent admits the spectral synthesis.*

Recall that the operator  $T$  is called  $m$ -accumulative ( $m$ -dissipative) if it has no accumulative (dissipative) extensions. It is well known that accumulative operator  $T$  is  $m$ -accumulative if and only if  $\rho(T) \neq \emptyset$ , or equivalently,  $\mathbb{C}_+ \subset \rho(T)$ . Now we are ready to prove our main result on the spectral synthesis.

**Theorem 6.12.** *Let  $B = B^*$  and let  $L_{C,D}(Q)$  be a complete accumulative (dissipative) operator. Then for any  $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$  the operator  $T(\lambda) := (L_{C,D}(Q) - \lambda)^{-1}$  exists and admits the spectral synthesis.*

*Proof.* For definiteness we confine ourself to the accumulative case. Since  $L_{C,D}(Q)$  is accumulative then by Lemma 6.9 condition (6.60) is satisfied and  $\text{Im } Q \leq 0$ . Hence Lemma (6.9) implies accumulativeness of  $L_{C,D}(0)$ . Hence by Lemma 6.10 condition (6.68) is satisfied. Therefore, it follows from Lemma 3.6 (see formula (3.69)) that the characteristic determinant  $\Delta(\cdot)$  is not identically zero. Thus,  $\rho(L_{C,D}(Q)) \neq \emptyset$ , and the operator  $L_{C,D}(Q)$  is  $m$ -accumulative. Therefore,  $\mathbb{C}_+ \subset \rho(L_{C,D}(Q))$  and operator  $T(\lambda) = (L_{C,D}(Q) - \lambda)^{-1}$  exists for all  $\lambda \in \mathbb{C}_+$ .



Since  $B = B^*$ , then, by Lemma 2.1, the adjoint operator  $L_{C,D}(Q)^*$  is  $L_{C,D}(Q)^* = L_{C_*,D_*}(Q^*)$  with appropriate  $n \times n$  matrices  $C_*$  and  $D_*$ . By Theorem 6.3,

$$(L_{C,D}(Q) - \bar{\lambda})^{-1} - (L_{C_*,D_*}(Q^*) - \bar{\lambda})^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \quad (6.73)$$

Further, combining Hilbert identity with Proposition 6.7 and taking into account property (6.30) of the classes  $\mathcal{S}_p$ , one gets

$$\begin{aligned} (L_{C,D}(Q) - \lambda)^{-1} - (L_{C,D}(Q) - \bar{\lambda})^{-1} &= 2\operatorname{Im} \lambda \cdot (L_{C,D}(Q) - \lambda)^{-1} \cdot (L_{C,D}(Q) - \bar{\lambda})^{-1} \\ &\in \mathcal{S}_1(\mathfrak{H}) \cdot \mathcal{S}_1(\mathfrak{H}) \subset \mathcal{S}_{1/2}(\mathfrak{H}) \subset \mathfrak{S}_1(\mathfrak{H}). \end{aligned} \quad (6.74)$$

In turn, combining last two relations we obtain

$$\begin{aligned} 2i \cdot \operatorname{Im} ((L_{C,D}(Q) - \lambda)^{-1}) &= (L_{C,D}(Q) - \lambda)^{-1} - (L_{C,D}(Q) - \bar{\lambda})^{-1} \\ &\quad + (L_{C,D}(Q) - \bar{\lambda})^{-1} - (L_{C_*,D_*}(Q^*) - \bar{\lambda})^{-1} \in \mathfrak{S}_1(\mathfrak{H}). \end{aligned} \quad (6.75)$$

Thus,  $T(\lambda)$ ,  $\lambda \in \mathbb{C}_+$ , is a compact accumulative operator with the imaginary part of trace class. Moreover,  $T(\lambda)$  is complete simultaneously with the operator  $L_{C,D}(Q)$ . Therefore, by the M.S. Brodskii theorem [11] (see also [47, Corollary 2.2], [53, Chapter 4.5]),  $T(\lambda)$  admits the spectral synthesis.  $\square$

**Remark 6.13.** *Emphasize, that Theorem 6.12 is stated only for the resolvent of  $L := L_{C,D}(Q)$ . In fact, the (unbounded) operator  $L$  itself does not admit the spectral synthesis. Indeed, the subspace  $\mathfrak{H}_a = \{y \in L^2[0,1] : y(x) = 0, x \in [0,a]\}$ ,  $a \in (0,1)$ , is invariant for  $L$ . However, it contains no eigenfunctions of  $L$  since, by the Cauchy uniqueness theorem, each solution of (1.2) vanishing at zero is identically zero.*

Further, we apply Theorem 2.4 and Lemma 6.10 to obtain more explicit result on spectral synthesis.

**Proposition 6.14.** *Let  $B = B^*$ ,  $\operatorname{Im} Q \leq 0$  and let the operator  $L_{C,D}(0)$  be accumulative. Assume also that for some  $C, R > 0$  and  $s \in \mathbb{Z}_+$*

$$|\Delta(-it)| \geq \frac{Ce^{\tau t}}{t^s}, \quad \tau = \sum_{b_j > 0} b_j, \quad t > R. \quad (6.76)$$

*Then the resolvent  $(L_{C,D}(Q) - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C}_+$ , admits the spectral synthesis.*

*Proof.* Since  $L_{C,D}(0)$  is accumulative, then, by Lemma 6.10,  $\det T_{-B}(C, D) \neq 0$ . Hence by Lemma 3.6 the estimate (2.16) holds for  $\Delta(\lambda)$  with  $z_1 = 1 + i$  and  $z_2 = -1 + i$  and  $s = 0$ . Estimate (6.76) yields (2.16) with  $z_3 = -i$ . Hence by Theorem 2.4 the system of root of functions of the operator  $L_{C,D}(Q)$  is complete in  $L^2([0,1]; \mathbb{C}^n)$ . Since  $\operatorname{Im} Q \leq 0$  then, by Lemma 6.9, the operator  $L_{C,D}(Q)$  is accumulative. Therefore, Theorem 6.12 implies the spectral synthesis.  $\square$

For  $2m \times 2m$  Dirac operator Proposition 6.14 reads as follows.

**Corollary 6.15.** *Let  $B = \operatorname{diag}(-I_m, I_m)$ ,  $\operatorname{Im} Q \leq 0$  and let the operator  $L_{C,D}(0)$  be accumulative. Assume also that for some  $C, R > 0$  and  $s \in \mathbb{Z}_+$*

$$|\Delta(-it)| \geq \frac{Ce^{mt}}{t^s}, \quad t > R. \quad (6.77)$$

*Then the resolvent  $(L_{C,D}(Q) - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C}_+$ , admits the spectral synthesis.*

Next, we clarify and complete Theorem 4.1 in the case of the accumulative (dissipative) operator  $L_{C,D}(0)$ . For definiteness we confine ourselves to the case of an accumulative operator only.

**Theorem 6.16.** *Let  $B = B^*$  and let the operator  $L_{C,D}(0)$  be accumulative. Assume also that one of the following conditions is satisfied:*

- (i)  $\det T_B(C, D) \neq 0$ ;
- (ii)  $Q$  is continuous at the endpoints 0 and 1 of the segment  $[0, 1]$  and

$$\sum_{\substack{b_j < 0 \\ b_k > 0}} \frac{\det T_B^{c_j \rightarrow c_k} b_k q_{kj}(0) - \det T_B^{d_k \rightarrow d_j} b_j q_{jk}(1)}{b_k - b_j} \neq 0. \quad (6.78)$$

Then the system of root functions of the operator  $L_{C,D}(Q)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ .

Moreover, if in addition,  $\operatorname{Im} Q \leq 0$ , then the resolvent  $(L_{C,D}(Q) - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C}_+$ , admits the spectral synthesis.

*Proof.* Since  $L_{C,D}(0)$  is accumulative, then, by Lemma 6.10,  $\det T_{-B}(C, D) \neq 0$ . Hence, if condition (i) is satisfied, then, by [45, Corollary 3.2], the system of root functions of the operator  $L_{C,D}(0)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^n)$ . Further, if  $\det T_B(C, D) = 0$ , then condition (ii) is satisfied. Clearly, condition (6.78) means that  $\omega_1(-i) \neq 0$ . Since  $\omega_0(i) = \det T_{-B}(C, D) \neq 0$ , it remains to apply Theorem 4.1.

If  $\operatorname{Im} Q \leq 0$  then, by Lemma 6.9, the operator  $L_{C,D}(Q)$  is accumulative. Since the system of root functions of  $L_{C,D}(Q)$  is complete in  $L^2([0, 1]; \mathbb{C}^n)$ , Theorem 6.12 implies the spectral synthesis.  $\square$

**Remark 6.17.** *In connection with Theorem 6.12 we briefly discuss the spectral synthesis for  $m$ -dissipative BVP for  $n$ th order ordinary differential equation (1.1) generated by  $n$  linearly independent boundary forms. First we note that if the resolvent set of a BVP for equation (1.1) is non-empty, then the resolvent  $R(\lambda)$  is trace class operator. For  $n = 2$ , i.e. for the operator  $-D^2 + q$ ,  $D := \frac{d}{dx}$ , this fact is implied by [27, III.10.4.3] since the Green function  $G(t, s)$  has essentially bounded derivative in mean with respect to  $t$ , for  $n \geq 3$  it is even  $C^1([0, 1] \times [0, 1])$ -kernel.*

*Note also, that if  $q \in L^2[0, 1]$  then  $\operatorname{dom}(-D^2 + q) \subset W^{2,2}[0, 1]$ , and hence  $R(\lambda) \in \mathcal{S}_{1/2}(L^2[0, 1])$  due to the properties of the embedding  $W^{2,2}[0, 1] \hookrightarrow L^2[0, a]$ . Alternatively for  $n = 2$  and  $q \in L^1[0, 1]$  one can adopt the proof of Theorem 6.3 and Proposition 6.7 taking the Dirichlet realization of  $-D^2$  in place of the periodic Dirac operator.*

*Further, by the Keldysh-Lidskii theorem [27, Theorem V.6.1], the dissipative operator  $R(\lambda)$ ,  $\lambda \in \mathbb{C}_-$ , is complete. To obtain the spectral synthesis it remains to apply the M.S. Brodskii theorem [11] (see also [53, IV.5]).*

## 7. APPLICATION TO THE TIMOSHENKO BEAM MODEL

Here we obtain some important geometric properties of the system of root functions for the dynamic generator of the Timoshenko beam model. Consider the following linear system of two coupled hyperbolic equations for  $t \geq 0$

$$I_\rho(x)\Phi_{tt} = K(x)(W_x - \Phi) + (EI(x)\Phi_x)_x - p_1(x)\Phi_t, \quad x \in [0, \ell], \quad (7.1)$$

$$\rho(x)W_{tt} = (K(x)(W_x - \Phi))_x - p_2(x)W_t, \quad x \in [0, \ell]. \quad (7.2)$$

The vibration of the Timoshenko beam of the length  $\ell$  clamped at the left end is governed by the system (7.1)–(7.2) subject to the following boundary conditions for  $t \geq 0$  [68]:

$$W(0, t) = \Phi(0, t) = 0, \quad (7.3)$$

$$(EI(x)\Phi_x(x, t) + \alpha_1\Phi_t(x, t) + \beta_1W_t(x, t))|_{x=\ell} = 0, \quad (7.4)$$

$$(K(x)(W_x(x, t) - \Phi(x, t)) + \alpha_2W_t(x, t) + \beta_2\Phi_t(x, t))|_{x=\ell} = 0. \quad (7.5)$$

Here  $W(x, t)$  is the lateral displacement at a point  $x$  and time  $t$ ,  $\Phi(x, t)$  is the bending angle at a point  $x$  and time  $t$ ,  $\rho(x)$  is a mass density,  $K(x)$  is the shear stiffness of a uniform cross-section,  $I_\rho(x)$  is the rotary inertia,  $EI(x)$  is the flexural rigidity at a point  $x$ ,  $p_1(x)$  and  $p_2(x)$  are locally distributed feedback functions,  $\alpha_j, \beta_j \in \mathbb{C}$ ,  $j \in \{1, 2\}$ . Boundary conditions at the right end contain as partial cases most of the known boundary conditions if  $\alpha_1, \alpha_2$  are allowed to be infinity.

Regarding the coefficients we assume that they satisfy the following general conditions:

$$\rho, I_\rho, K, EI \in C[0, \ell], \quad p_1, p_2 \in L^1[0, \ell], \quad (7.6)$$

$$0 < C_1 \leq \rho(x), I_\rho(x), K(x), EI(x) \leq C_2, \quad x \in [0, \ell]. \quad (7.7)$$

The energy space associated with the problem (7.1)–(7.5) is

$$\mathfrak{H} := \tilde{H}_0^1[0, \ell] \times L^2[0, \ell] \times \tilde{H}_0^1[0, \ell] \times L^2[0, \ell], \quad (7.8)$$

where  $\tilde{H}_0^1[0, \ell] := \{f \in W^{1,2}[0, \ell] : f(0) = 0\}$ . The norm in the energy space is defined as follows:

$$\|y\|_{\mathfrak{H}}^2 = \int_0^\ell (EI|y_1'|^2 + I_\rho|y_2|^2 + K|y_3' - y_1|^2 + \rho|y_4|^2) dx, \quad y = \text{col}(y_1, y_2, y_3, y_4). \quad (7.9)$$

The problem (7.1)–(7.5) can be rewritten as

$$y_t = i\mathcal{L}y, \quad y(x, t)|_{t=0} = y_0(x), \quad (7.10)$$

where  $y$  and  $\mathcal{L}$  are given by

$$y = \begin{pmatrix} \Phi(x, t) \\ \Phi_t(x, t) \\ W(x, t) \\ W_t(x, t) \end{pmatrix}, \quad \mathcal{L} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \frac{1}{i} \begin{pmatrix} y_2 \\ \frac{1}{I_\rho(x)} \left( K(x)(y_3' - y_1) + (EI(x)y_1')' - p_1(x)y_2 \right) \\ y_4 \\ \frac{1}{\rho(x)} \left( (K(x)(y_3' - y_1))' - p_2(x)y_4 \right) \end{pmatrix} \quad (7.11)$$

on the domain

$$\begin{aligned} \text{dom}(\mathcal{L}) = \Big\{ & y = \text{col}(y_1, y_2, y_3, y_4) : y_1, y_2, y_3, y_4 \in \tilde{H}_0^1[0, \ell], \\ & EI \cdot y_1' \in AC[0, \ell], \quad (EI \cdot y_1')' - p_1 y_2 \in L^2[0, \ell], \\ & K \cdot (y_3' - y_1) \in AC[0, \ell], \quad (K \cdot (y_3' - y_1))' - p_2 y_4 \in L^2[0, \ell], \\ & (EI \cdot y_1')(\ell) + \alpha_1 y_2(\ell) + \beta_1 y_4(\ell) = 0, \\ & (K \cdot (y_3' - y_1))(\ell) + \alpha_2 y_4(\ell) + \beta_2 y_2(\ell) = 0 \Big\}. \end{aligned} \quad (7.12)$$

Timoshenko beam model is investigated in numerous papers (see [67, 68, 34, 62, 63, 74, 73, 72] and the references therein). A number of stability, controllability, and optimization problems were studied. Note also that the general model (7.1)–(7.5) of spatially non-homogenous Timoshenko beam with both boundary and locally distributed damping covers the cases studied by many authors. Geometric properties of the system of root functions of the operator  $\mathcal{L}$  play important role in investigation of different properties of the problem (7.1)–(7.5).

Below we establish completeness and the Riesz basis property with parentheses of the operator  $\mathcal{L}$ , without analyzing its spectrum. For convenience we impose the following additional algebraic assumption on  $\mathcal{L}$ :

$$\nu(x) := \frac{EI(x)\rho(x)}{K(x)I_\rho(x)} = \text{const}, \quad x \in [0, \ell], \quad (7.13)$$

Clearly, (7.13) is satisfied whenever  $I_\rho(x) = R\rho(x)$ , where  $R = \text{const}$  is a cross-sectional area of the beam,  $EI$  and  $K$  are constant functions, while  $\rho \in AC[0, \ell]$  and is arbitrary positive (cf. condition (7.19)). Our approach to the spectral properties of the operator  $\mathcal{L}$  is based on the similarity reduction of  $\mathcal{L}$  to a special  $4 \times 4$  Dirac-type operator. To state the result we need some additional preparations.

Let  $\gamma(\cdot)$  be given by

$$\sqrt{\frac{I_\rho(x)}{EI(x)}} = b_1\gamma(x), \quad \text{where } b_1 > 0 \quad \text{and} \quad \int_0^\ell \gamma(x)dx = 1. \quad (7.14)$$

Conditions (7.6) and (7.7) imply together that  $\gamma \in C[0, \ell]$  and is positive. Further, in view of (7.13) we have

$$\sqrt{\frac{\rho(x)}{K(x)}} = b_2\gamma(x), \quad \text{where } b_2 > 0. \quad (7.15)$$

Let

$$B := \text{diag}(-b_1, b_1, -b_2, b_2). \quad (7.16)$$

$$\Theta(x) := -2i \text{diag}(I_\rho(x), I_\rho(x), \rho(x), \rho(x)), \quad (7.17)$$

$$h_1(x) := \sqrt{EI(x)I_\rho(x)}, \quad h_2(x) := \sqrt{K(x)\rho(x)}. \quad (7.18)$$

In the sequel we assume that

$$h_1, h_2 \in AC[0, \ell]. \quad (7.19)$$

Therefore, according to (7.6)–(7.7) the following matrix function is well-defined:

$$\widehat{Q}(x) := \Theta^{-1}(x) \begin{pmatrix} p_1 + h'_1 & p_1 - h'_1 & h_2 & -h_2 \\ p_1 + h'_1 & p_1 - h'_1 & h_2 & -h_2 \\ -h_2 & -h_2 & p_2 + h'_2 & p_2 - h'_2 \\ h_2 & h_2 & p_2 + h'_2 & p_2 - h'_2 \end{pmatrix}. \quad (7.20)$$

Next, we set

$$t(x) = \int_0^x \gamma(s)ds, \quad x \in [0, \ell]. \quad (7.21)$$

Since  $\gamma \in C[0, \ell]$  and is positive, the function  $t(\cdot)$  strictly increases on  $[0, \ell]$ ,  $t(\cdot) \in C^1[0, \ell]$ , and due to (7.14)  $t(\ell) = 1$ . Hence, the inverse function  $x(\cdot) := t^{-1}(\cdot)$  is well defined, strictly increasing on  $[0, 1]$ , and  $x(\cdot) \in C^1[0, 1]$ . Next, we put

$$Q(t) := \widehat{Q}(x(t)) =: (q_{jk}(t))_{j,k=1}^4, \quad t \in [0, 1]. \quad (7.22)$$

Finally, let

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 - h_1(\ell) & \alpha_1 + h_1(\ell) & \beta_1 & \beta_1 \\ 0 & 0 & 0 & 0 \\ \beta_2 & \beta_2 & \alpha_2 - h_2(\ell) & \alpha_2 + h_2(\ell) \end{pmatrix}. \quad (7.23)$$

**Proposition 7.1.** *Let functions  $\rho, I_\rho, K, EI, p_1, p_2, h_1, h_2$  satisfy conditions (7.6), (7.7), (7.13) and (7.19). Then the operator  $\mathcal{L}$  is similar to the  $4 \times 4$  Dirac-type operator  $L := L_{C,D}(Q)$  with the matrices  $B, C, D, Q(\cdot)$  given by (7.16), (7.23) and (7.22).*

*Proof.* Introduce the following operator

$$Uy = \text{col}(EI(x)y'_1, y_2, K(x)(y'_3 - y_1), y_4), \quad y = \text{col}(y_1, y_2, y_3, y_4), \quad (7.24)$$

that maps the Hilbert space  $\mathfrak{H}$  given by (7.8) into  $L^2([0, \ell]; \mathbb{C}^4)$ . Since  $\frac{d}{dx}$  isometrically maps

$$\tilde{H}_0^1[0, \ell] = \{f \in W^{1,2}[0, \ell] : f(0) = 0\} \quad (7.25)$$

onto  $L^2[0, \ell]$ , it follows from conditions (7.7) that the operator  $U$  is bounded with bounded inverse. It is easy to check that for  $y = \text{col}(y_1, y_2, y_3, y_4)$

$$\mathcal{L}U^{-1}y = \frac{1}{i} \begin{pmatrix} y_2 \\ \frac{1}{I_\rho}(y'_1 - p_1y_2 + y_3) \\ y_4 \\ \frac{1}{\rho}(y'_3 - p_2y_4) \end{pmatrix}, \quad \tilde{L}y := U\mathcal{L}U^{-1}y = \frac{1}{i} \begin{pmatrix} EI \cdot y'_2 \\ \frac{1}{I_\rho}(y'_1 - p_1y_2 + y_3) \\ K \cdot (y'_4 - y_2) \\ \frac{1}{\rho}(y'_3 - p_2y_4) \end{pmatrix}, \quad (7.26)$$

and

$$\begin{aligned} \text{dom}(\tilde{L}) = U \text{dom}(\mathcal{L}) &= \{y = \text{col}(y_1, y_2, y_3, y_4) \in W^{1,1}([0, \ell]; \mathbb{C}^4) : \\ &\tilde{L}y \in L^2([0, \ell]; \mathbb{C}^4), \quad y_2(0) = y_4(0) = 0, \\ &y_1(\ell) + \alpha_1 y_2(\ell) + \beta_1 y_4(\ell) = 0, \quad y_3(\ell) + \alpha_2 y_4(\ell) + \beta_2 y_2(\ell) = 0\}. \end{aligned} \quad (7.27)$$

Thus, the operator  $\mathcal{L}$  is similar to the operator  $\tilde{L}$ ,

$$\tilde{L}y = -i\tilde{B}(x)y' + \tilde{Q}(x)y \quad (7.28)$$

with the domain  $\text{dom}(\tilde{L})$  given by (7.27), and the matrix functions  $\tilde{B}(\cdot), \tilde{Q}(\cdot)$ , given by

$$\tilde{B}(x) := \begin{pmatrix} 0 & EI(x) & 0 & 0 \\ \frac{1}{I_\rho(x)} & 0 & 0 & 0 \\ 0 & 0 & 0 & K(x) \\ 0 & 0 & \frac{1}{\rho(x)} & 0 \end{pmatrix}, \quad \tilde{Q}(x) := i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{p_1(x)}{I_\rho(x)} & -\frac{1}{I_\rho(x)} & 0 \\ 0 & K(x) & 0 & 0 \\ 0 & 0 & 0 & \frac{p_2(x)}{\rho(x)} \end{pmatrix}. \quad (7.29)$$

Note, that  $\tilde{Q} \in L^1([0, \ell]; \mathbb{C}^{4 \times 4})$  in view of conditions (7.6)–(7.7). Next we diagonalize the matrix  $\tilde{B}(\cdot)$ . Namely, setting

$$\tilde{U}(x) := \begin{pmatrix} -h_1(x) & h_1(x) & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -h_2(x) & h_2(x) \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (7.30)$$

and noting that

$$\tilde{U}^{-1}(x) = \frac{1}{2} \begin{pmatrix} -\frac{1}{h_1(x)} & 1 & 0 & 0 \\ \frac{1}{h_1(x)} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{h_2(x)} & 1 \\ 0 & 0 & \frac{1}{h_2(x)} & 1 \end{pmatrix}, \quad (7.31)$$

we easily get after straightforward calculations

$$\tilde{U}^{-1}(x)\tilde{B}(x)\tilde{U}(x) = \text{diag} \left( -\sqrt{\frac{EI(x)}{I_\rho(x)}}, \sqrt{\frac{EI(x)}{I_\rho(x)}}, -\sqrt{\frac{K(x)}{\rho(x)}}, \sqrt{\frac{K(x)}{\rho(x)}} \right) = \frac{1}{\gamma(x)}B^{-1}, \quad (7.32)$$

Here we have used definition (7.18) of  $h_1$ ,  $h_2$ , and definitions (7.14) and (7.15) of  $b_1$ ,  $b_2$ , and  $\gamma(x)$ , respectively. Further, note that

$$\tilde{U}(\cdot) \in W^{1,1}([0, \ell]; \mathbb{C}^{4 \times 4}) \quad \text{and} \quad \hat{Q} \in L^1([0, \ell]; \mathbb{C}^{4 \times 4}) \quad (7.33)$$

in view of (7.6), (7.7) and (7.19), where  $\hat{Q}(\cdot)$  is given by (7.20) and (7.17). Hence, it is easily seen that

$$\tilde{U}^{-1}(x)\tilde{Q}(x)\tilde{U}(x) - i\tilde{U}^{-1}(x)\tilde{B}(x)\tilde{U}'(x) = \hat{Q}(x), \quad x \in [0, \ell]. \quad (7.34)$$

Introducing the operator  $\tilde{U} : y \rightarrow \tilde{U}(x)y$  in  $L^2([0, \ell]; \mathbb{C}^4)$  and taking into account (7.32) and (7.34) we obtain that for any  $y \in W^{1,1}([0, \ell]; \mathbb{C}^4)$  and satisfying  $\tilde{U}y \in \text{dom}(\tilde{L})$

$$\hat{L}y := \tilde{U}^{-1}\tilde{L}\tilde{U}y = -i\gamma(x)^{-1}B^{-1}y' + \hat{Q}(x)y. \quad (7.35)$$

Next, taking into account formulas (7.23) and (7.30) for matrices  $C$ ,  $D$ , and  $\tilde{U}(\cdot)$ , respectively, we derive

$$\text{dom}(\hat{L}) = \{y \in W^{1,1}([0, \ell]; \mathbb{C}^4) : \hat{L}y \in L^2([0, \ell]; \mathbb{C}^4), Cy(0) + Dy(\ell) = 0\}. \quad (7.36)$$

Finally, we apply similarity transformation  $S$  that realizes the change of variable  $x = x(t)$ ,

$$S : L^2([0, \ell]; \mathbb{C}^4) \rightarrow L^2([0, 1]; \mathbb{C}^4), \quad (Sf)(t) = f(x(t)), \quad t \in [0, 1]. \quad (7.37)$$

Since both  $t(\cdot)$  and  $x(\cdot)$  are strictly increasing and continuously differentiable, the following implications hold

$$f(\cdot) \in W^{1,1}([0, \ell]; \mathbb{C}^4) \Rightarrow f(x(\cdot)) \in W^{1,1}([0, 1]; \mathbb{C}^4), \quad (7.38)$$

$$g(\cdot) \in W^{1,1}([0, 1]; \mathbb{C}^4) \Rightarrow g(t(\cdot)) \in W^{1,1}([0, \ell]; \mathbb{C}^4). \quad (7.39)$$

Hence (7.36) and (1.5) implies  $\text{dom}(L) = S \text{dom}(\hat{L})$ . Next, it follows from (7.21) that  $t'(x) = \gamma(x)$ ,  $x \in [0, \ell]$ . Hence for  $f \in \text{dom}(L)$  and  $x \in [0, \ell]$  one has

$$\begin{aligned} (\hat{L}S^{-1}f)(x) &= -i\gamma(x)^{-1}B^{-1}\frac{d}{dx}[f(t(x))] + \hat{Q}(x)f(t(x)) \\ &= -iB^{-1}f'(t(x)) + \hat{Q}(x)f(t(x)), \end{aligned} \quad (7.40)$$

which directly implies that  $L = S\hat{L}S^{-1}$ . Combining this identity with (7.26) and (7.35) one concludes that  $\mathcal{L}$  is similar to  $L = L_{C,D}(Q)$ .  $\square$

**Remark 7.2.** Proposition 7.1 remains valid if we replace condition (7.6) by the weaker assumption  $\rho, I_\rho, K, EI \in L^\infty[0, \ell]$  and assume in addition that the inverse function  $x(\cdot) = t^{-1}(\cdot)$  is absolutely continuous. Otherwise implication (7.38) fails, since in general the inverse function of absolutely continuous function is not necessarily absolutely continuous. For instance, the function  $h(x) := x + C(x)$ ,  $x \in [0, 1]$ , where  $C(\cdot)$  is the Cantor function, strictly increases and is not absolutely continuous. At the same time, the inverse function is absolutely continuous.

Applying [45, Corollary 3.2] and Theorem 5.6 to the operator  $L$  we obtain the following result.

**Theorem 7.3.** Let conditions (7.6), (7.7), (7.13), (7.19) be satisfied and let also

$$(\alpha_1 + h_1(\ell))(\alpha_2 + h_2(\ell)) \neq \beta_1\beta_2 \quad \text{and} \quad (\alpha_1 - h_1(\ell))(\alpha_2 - h_2(\ell)) \neq \beta_1\beta_2. \quad (7.41)$$

(i) Then the system of root functions of  $\mathcal{L}$  is complete and minimal in  $\mathfrak{H}$ .

(ii) Assume in addition that

$$p_1, p_2 \in L^\infty[0, \ell], \quad h_1, h_2 \in \text{Lip}_1[0, \ell] \quad \text{and} \quad \beta_1 = \beta_2 = 0. \quad (7.42)$$

Then the system of root functions of the operator  $\mathcal{L}$  forms a Riesz basis with parentheses in  $\mathfrak{H}$ .

*Proof. (i)* Consider the operator  $L_{C,D}(Q)$  defined in Proposition 7.1. Combining expressions (7.16) and (7.23) for the matrices  $B, C, D$  with definition of  $T_A(C, D)$  yields

$$\det T_B(C, D) = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_1 + h_1(\ell) & 0 & \beta_1 \\ 0 & 0 & 1 & 0 \\ 0 & \beta_2 & 0 & \alpha_2 + h_2(\ell) \end{pmatrix} = (\alpha_1 + h_1(\ell))(\alpha_2 + h_2(\ell)) - \beta_1\beta_2. \quad (7.43)$$

Similarly one gets

$$\det T_{-B}(C, D) = (\alpha_1 - h_1(\ell))(\alpha_2 - h_2(\ell)) - \beta_1\beta_2. \quad (7.44)$$

Conditions (7.41) implies  $\det T_B(C, D) \neq 0$  and  $\det T_{-B}(C, D) \neq 0$ . Therefore, by [45, Corollary 3.2], the system of root functions of the operator  $L_{C,D}(Q)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^4)$ . Since, by Proposition 7.1,  $\mathcal{L}$  is similar to the operator  $L_{C,D}(Q)$ , the system of root functions of the operator  $\mathcal{L}$  is complete and minimal in  $\mathfrak{H}$ .

(ii) Again consider the operator  $L_{C,D}(Q)$  defined in Proposition 7.1. Since  $\beta_1 = \beta_2 = 0$  and (7.41) is fulfilled, then according to (7.16) and (7.23) the matrices  $B, C, D$  have the block structure described in (5.36)–(5.38) with  $r = 2$ . Moreover, (7.42) implies  $Q \in L^\infty([0, 1]; \mathbb{C}^{4 \times 4})$ . Therefore, combining Theorem 5.6 with Proposition 7.1 yields the statement.  $\square$

Applying Corollary 4.9 we can improve Theorem 7.3(i) assuming that  $\widehat{Q}(\cdot)$  is continuous at the endpoints 0,  $\ell$ . For simplicity we assume that  $\beta_1 = \beta_2 = 0$ .

**Theorem 7.4.** *Let the functions  $\rho, I_\rho, K, EI, p_1, p_2, h_1, h_2$  satisfy conditions (7.6), (7.7), (7.13) and (7.19). Let also the functions  $p_1, p_2, h'_1, h'_2$  be continuous at the endpoints 0 and  $\ell$ . Assume in addition that  $\beta_1 = \beta_2 = 0$  and the following assumptions are fulfilled:*

- (i)  $|\alpha_1 - h_1(\ell)| + |\alpha_2 - h_2(\ell)| \neq 0$  and  $|\alpha_1 + h_1(\ell)| + |\alpha_2 + h_2(\ell)| \neq 0$ ;
- (ii) for each  $j \in \{1, 2\}$  one of the following conditions is satisfied:
  - (a)  $\alpha_j^2 \neq h_j^2(\ell)$ ;
  - (b)  $\alpha_j = h_j(\ell)$  and  $h'_j(\ell) \neq -p_j(\ell)$ ;
  - (c)  $\alpha_j = -h_j(\ell)$  and  $h'_j(\ell) \neq p_j(\ell)$ .

Then the system of root functions of  $\mathcal{L}$  is complete and minimal in  $\mathfrak{H}$ .

*Proof.* Consider the operator  $L_{C,D}(Q)$  defined in Proposition 7.1. Since  $\rho, I_\rho \in C[0, \ell]$  and  $p_1, p_2, h'_1, h'_2$  are continuous at the endpoints 0 and  $\ell$ , it follows from (7.17)–(7.22) that the matrix function  $Q(\cdot)$  is continuous at the endpoints 0 and 1. Since  $\beta_1 = \beta_2 = 0$ , the block matrix representations (7.16) and (7.23) of the matrices  $B, C, D$ , allow to apply Corollary 4.9 and Lemma 4.10. Let us verify conditions (4.20)–(4.23) of Lemma 4.10. First, comparing (4.11) with (7.23) yields

$$d_1 = \alpha_1 - h_1(\ell), \quad d_2 = \alpha_1 + h_1(\ell), \quad (7.45)$$

$$d_3 = \alpha_2 - h_2(\ell), \quad d_4 = \alpha_2 + h_2(\ell). \quad (7.46)$$



Therefore, condition (4.20) is always satisfied, since  $h_j(\ell) \neq 0$ ,  $j \in \{1, 2\}$ , while condition (4.21) is equivalent to the condition (i) of the theorem. Further, it follows from (7.20) and (7.22) that

$$q_{12}(1) = \frac{p_1(\ell) - h'_1(\ell)}{-2iI_\rho(\ell)}, \quad q_{21}(1) = \frac{p_1(\ell) + h'_1(\ell)}{-2iI_\rho(\ell)}, \quad (7.47)$$

$$q_{34}(1) = \frac{p_2(\ell) - h'_2(\ell)}{-2i\rho(\ell)}, \quad q_{43}(1) = \frac{p_2(\ell) + h'_2(\ell)}{-2i\rho(\ell)}. \quad (7.48)$$

Hence, conditions (4.22) and (4.23) are equivalent to the conditions (a)-(c) of the theorem for  $j = 1$  and  $j = 2$ , respectively. Therefore, by Lemma 4.10, condition (4.12) is satisfied and, by Corollary 4.9, the system of root functions of the operator  $L_{C,D}(Q)$  is complete and minimal in  $L^2([0, 1]; \mathbb{C}^4)$ . Therefore, Proposition 7.1 completes the proof.  $\square$

**Remark 7.5.** *The main results remain also valid if the function  $\nu(\cdot)$  given by (7.13) satisfies  $\nu(x) \neq 1$  for  $x \in [0, \ell]$ .*

**Remark 7.6.** (i) *In connection with Theorem 7.3 we mention the paper [62] where the operator  $\mathcal{L}$  was investigated under the following assumptions on the parameters of the model:*

$$EI, K \in W^{3,2}[0, \ell], \quad \rho, I_\rho \in W^{4,2}[0, \ell], \quad p_1 = p_2 = 0, \quad \beta_1 = \beta_2 = 0, \quad (7.49)$$

*but without algebraic assumption (7.13). The completeness of the root functions was stated in [62] under the condition (7.41) and the additional assumption*

$$I_\rho(x)K(x) \neq \rho(x)EI(x), \quad x \in [0, \ell], \quad (7.50)$$

*which in our notations means that  $\nu(x) \neq 1$ ,  $x \in [0, \ell]$ . Unfortunately, the proof of the completeness in [62] fails because of the incorrect application of the Keldysh theorem. Namely, the representation  $\mathcal{L}^{-1} = \mathcal{L}_{00}^{-1}(I_{\mathcal{H}} + T)$  used in [62], where  $T$  is of finite rank bounded operator and  $\mathcal{L}_{00} = \mathcal{L}_{00}^*$ , fails since it leads to the inclusion  $\text{dom}(\mathcal{L}) \subset \text{dom}(\mathcal{L}_{00})$ , which holds if only if  $\mathcal{L} = \mathcal{L}_{00}$ .*

*Moreover, under conditions (7.49), (7.50) and (7.41) the Riesz basis property for the system of root functions of  $\mathcal{L}$  was stated in [62]. The proof is based on the fact that under the above restrictions the eigenvalues of  $\mathcal{L}$  are asymptotically simple and separated. However, it is not the case. For instance, if  $K \equiv EI \equiv \rho \equiv 1$ ,  $I_\rho \equiv 4$ ,  $\alpha_1 = 5/2$  and  $\alpha_2 = 13/12$ , then according to [62, Theorem 4.2] the sequence of the eigenvalues of  $\mathcal{L}$  splits into two families*

$$\lambda_n^{(1)} = \frac{\pi n}{2} + \frac{i}{2} \ln 3 + O(n^{-1}) \quad \text{and} \quad \lambda_n^{(2)} = \pi n + \frac{i}{2} \ln 3 + O(n^{-1}), \quad n \in \mathbb{Z} \setminus \{0\}. \quad (7.51)$$

*Clearly, in this case the sequence of the eigenvalues of  $\mathcal{L}$  is not asymptotically simple and separated. Note, however, that according to Theorem 7.3(ii) the system of root functions of the operator  $\mathcal{L}$  always forms a Riesz basis with parentheses under the restrictions (7.6), (7.7), (7.13), (7.19), (7.41) and (7.42).*

(ii) *In connection with Theorem 7.3 we also mention the paper [74]. In this paper the operator  $\mathcal{L}$  was investigated under the following stronger assumptions on the parameters of the model:*

$$EI, K, \rho, I_\rho \text{ are constant}, \quad p_1 = p_2 = 0, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0, \quad 4\alpha_1\alpha_2 \geq (\beta_1 + \beta_2)^2. \quad (7.52)$$

*The last condition in (7.52) ensures the dissipativity of the operator  $\mathcal{L}$ . The completeness of the system of root functions of the operator  $\mathcal{L}$  was proved in [74] under the restrictions (7.52) and (7.41). So, our Theorem 7.3(i) generalizes this result to a broader class of boundary conditions and improves it in the dissipative case. Note also that under additional assumptions, guarantying that the eigenvalues of  $\mathcal{L}$  are asymptotically simple and separated, it was proved*

in [74] that the root functions of  $\mathcal{L}$  contains the Riesz basis. Moreover, this fact was applied to show the exponential stability of the problem (7.1)–(7.5).

**Acknowledgments.** We are indebted to D. Yakubovich for the reformulation of the condition (a) of Theorem 4.1 mentioned in Remark 4.2. We are also indebted to A. Shkalikov for useful remarks helping us to improve the exposition.

## REFERENCES

- [1] A. V. Agibalova, M. M. Malamud and L. L. Oridoroga, On the completeness of general boundary value problems for  $2 \times 2$  first-order systems of ordinary differential equations. *Methods Funct. Anal. and Topology* (1) **18** (2012), 4–18.
- [2] F.V. Atkinson, Discrete and continuous boundary problems, Academic Press, New York, 1964.
- [3] A. Baranov, Y. Belov and A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, *Adv. Math.* **235** (2013), pp. 525–554.
- [4] A. Baranov, Y. Belov, A. Borichev and D. Yakubovich, Recent developments in spectral synthesis for exponential systems and for non-self-adjoint operators, arXiv:1212.6014 (Submitted on 25 Dec 2012), to appear in *Recent Trends in Analysis*, Proceedings of the conference in honor of N. Nikolski (Bordeaux, 2011), Theta Foundation, Bucharest, 2013.
- [5] A. Baranov and D. Yakubovich, Completeness and spectral synthesis of nonselfadjoint one-dimensional perturbations of selfadjoint operators, arXiv:1212.5965 (Submitted on 24 Dec 2012).
- [6] A. Baranov and D. Yakubovich, One-dimensional perturbations of unbounded selfadjoint operators with empty spectrum, arXiv:1304.5800 (Submitted on 21 Apr 2013).
- [7] A. G. Baskakov, A. V. Derbushev and A. O. Shcherbakov, The method of similar operators in the spectral analysis of non-self-adjoint Dirac operators with non-smooth potentials. *Izv. Math.* (3) **75** (2011), 445–469.
- [8] G. D. Birkhoff, On the asymptotic character of the solution of the certain linear differential equations containing parameter. *Trans. Amer. Math. Soc.* (2) **9** (1908), 219–231.
- [9] G. D. Birkhoff, Boundary value and expansion problems of ordinary linear differential equations. *Trans. Amer. Math. Soc.* **9** (1908), 373–395.
- [10] G. D. Birkhoff and R. E. Langer, The boundary problems and developments associated with a system of ordinary differential equations of the first order. *Proc. Amer. Acad. Arts Sci.* **58** (1923), 49–128.
- [11] M.S. Brodskii, On operators with trace class imaginary components, *Acta Sci. Math. Szeged*, **27** (3–4) (1966), 147–155.
- [12] P. Djakov and B. Mityagin, Instability zones of 1D periodic Schrödinger and Dirac operators, *Russian Math. Surveys* **61** (4) (2006), 663–766.
- [13] P. Djakov and B. Mityagin, Bari-Markus property for Riesz projections of 1D periodic Dirac operators. *Math. Nachr.* (3) **283** (2010), 443–462.
- [14] P. Djakov and B. Mityagin, Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions. *Indiana Univ. Math. J.* (1) **61** (2012), 359–398.
- [15] P. Djakov and B. Mityagin, Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials, *Math. Ann.*, **351** (3) (2011), 509–540.
- [16] P. Djakov and B. Mityagin, 1D Dirac operators with special periodic potentials. *Bull. Polish Acad. Sci. Math.* (1) **60** (2012), 59–75.
- [17] P. Djakov and B. Mityagin, Equiconvergence of spectral decompositions of 1D Dirac operators with regular boundary conditions. *J. Approximation Theory* (7) **164** (2012), 879–927.
- [18] P. Djakov and B. Mityagin, Criteria for existence of Riesz bases consisting of root functions of Hill and 1D Dirac operators. *J. Funct. Anal.* (8) **263** (2012), 2300–2332.
- [19] P. Djakov and B. Mityagin, Riesz bases consisting of root functions of 1D Dirac operators. *Proc. Amer. Math. Soc.* (4) **141** (2013), 1361–1375.
- [20] N. Dunford, A Survey of the Theory of Spectral Operators, *Bull. Am. Math. Soc.* **64** (1958), 217–274.
- [21] N. Dunford and J. Schwartz, Linear Operators, Part I, General Theory, Wiley, New York, 1958.
- [22] N. Dunford and J. Schwartz, *Linear Operators, Part III, Spectral Operators*. Wiley, New York 1971.
- [23] F. Gesztesy, M. Malamud, M. Mitrea and S. Naboko, Generalized Polar Decompositions for Closed Operators in Hilbert Spaces and Some Applications, *Integral Equat. Oper. Theor.* **64** (2009), 83–113.
- [24] F. Gesztesy and V. Tkachenko, A criterion for Hill operators to be spectral operators of scalar type. *J. Analyse Math.* **107** (2009), 287–353.

- [25] F. Gesztesy and V. Tkachenko, A Schauder and Riesz basis criterion for non-selfadjoint Schrödinger operators with periodic and anti-periodic boundary conditions. *J. Diff. Equat.* (2) **253** (2012), 400–437.
- [26] Yu. P. Ginzburg, The almost invariant spectral properties of contractions and the multiplicative properties of analytic operator-functions. *Funct. Anal. Appl.* (3) **5** (1971), 197–205.
- [27] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space*. Transl. Math. Monographs, **18**, Amer. Math. Soc., Providence, R.I. 1969.
- [28] G. M. Gubreev, On the spectral decomposition of finite-dimensional perturbations of dissipative Volterra operators. *Tr. Mosc. Mat. Obs.* **64** (2003) 90–140; translation in *Trans. Moscow Math. Soc.* 2003, 79–126.
- [29] S. Hassi and L. Oridoroga, Theorem of Completeness for a Dirac-Type Operator with Generalized  $\lambda$ -Depending Boundary Conditions. *Integral Equat. Oper. Theor.* **64** (2009), 357–379.
- [30] M. V. Keldysh, On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations. *Doklady Akad. Nauk SSSR (N.S.)* (1) **77** (1951), 11–14 (in Russian).
- [31] G. M. Keselman, On the unconditional convergence of eigenfunction expansions of certain differential operators. *Izv. Vyssh. Uchebn. Zaved. Mat.* (2) **39** (1964), 82–93 (in Russian).
- [32] A.P. Khromov, Generating functions of Volterra operators *Mat. Sb. (N.S.)* **102(144)**, (3) (1977), 457–472 (in Russian).
- [33] A. P. Khromov, Finite dimensional perturbations of Volterra operators. *J. Math. Sci.* (N. Y.) (5) **138** (2006), 5893–6066.
- [34] J. U. Kim and Y. Renardy, Boundary Control of the Timoshenko Beam. *SIAM J. Control and Optimization*, (6) **25** (1987), 1417–1429.
- [35] A. G. Kostyuchenko and A. A. Shkalikov, Summability of expansions in eigenfunctions of differential operators and of convolution operators. *Funct. Anal. Appl.* (4) **12** (1978), 262–276.
- [36] B. Ya. Levin, *Lectures on Entire Functions*, Transl. Math. Monographs, **150**, Amer. Math. Soc., Providence, R.I. 1996 (in collaboration with Yu. Lyubarskii, M. Sodin, and V. Tkachenko).
- [37] B. M. Levitan and I. S. Sargsyan, *Sturm-Liouville And Dirac Operators*. Kluwer, Dordrecht 1991.
- [38] A. A. Lunyov and M. M. Malamud, On the completeness of the root vectors for first order systems. *Dokl. Math.* (3) **88** (2013), 678–683.
- [39] Yu. I. Lyubarskii and V. A. Tkachenko, System  $\{e^{\alpha n x} \sin nx\}$ . *Funct. Anal. Appl.* (2) **18** (1984), 144–146.
- [40] A. S. Makin, On Summability of Spectral Expansions Corresponding to the Sturm-Liouville Operator. *Inter. J. Math. and Math. Sci.* **2012** (2012), ID 843562, 13 p.
- [41] M. M. Malamud, Questions of uniqueness in inverse problems for systems of differential equations on a finite interval. *Trans. Moscow Math. Soc.* **60** 1999, 173–224.
- [42] M. M. Malamud, On the completeness of a system of root vectors of the Sturm-Liouville operator with general boundary conditions. *Funct. Anal. Appl.* (3) **42** (2008), 198–204.
- [43] M. M. Malamud and L. L. Oridoroga, Completeness theorems for systems of differential equations. *Funct. Anal. Appl.* (4) **34** (2000), 308–310.
- [44] M.M. Malamud and L.L. Oridoroga, On the completeness of the root vectors of first order systems, *Dokl. Math.*, **82** (3) (2010), 899–905.
- [45] M. M. Malamud and L. L. Oridoroga, On the completeness of root subspaces of boundary value problems for first order systems of ordinary differential equations. *J. Funct. Anal.* **263** (2012), 1939–1980; arXiv:0320048.
- [46] V. A. Marchenko, *Sturm-Liouville operators and applications*. Operator Theory: Advances and Appl. **22**, Birkhäuser Verlag, Basel 1986.
- [47] A.S. Markus, The problem of spectral synthesis for operators with point spectrum, *Mathematics of the USSR-Izvestiya*, **4** (3) (1970), 670–696.
- [48] A. S. Markus, *An Introduction to the Spectral Theory of Polynomial Operator Pencils*. Shtiintsa, Chisinau, 1986 (in Russian).
- [49] A. S. Markus and V. I. Matsaev, Comparison Theorems for Spectra of Linear Operators and Spectral Asymptotics. *Tr. Mosk. Mat. Obs.* **45** (1982), 133–181.
- [50] V. P. Mikhailov, On Riesz bases in  $L^2(0, 1)$ . *Dokl. Akad. Nauk SSSR* **144** (1962), 981–984 (in Russian).
- [51] B. Mityagin, Spectral expansions of one-dimensional periodic Dirac operators, *Dyn. Partial Differ. Equ.* **1** (2004), 125–191.
- [52] M. A. Naimark, *Linear differential operators, Part I*. Frederick Ungar Publishing Co., New York 1967.
- [53] N.K. Nikolski, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.
- [54] S. P. Novikov, S. V. Manakov, L. P. Pitaevskij and V. E. Zakharov, *Theory of solitons. The inverse scattering method*. Springer-Verlag 1984.

- [55] V. S. Rykhlov, Completeness of eigenfunctions of one class of pencils of differential operators with constant coefficients. *Russian Mathematics (Iz VUZ)* (6) **53** (2009), 33–43.
- [56] A. A. Shkalikov, The completeness of the eigen- and associated functions of an ordinary differential operator with nonregular splitting boundary conditions. *Funct. Anal. Appl.* (4) **10** (1976), 305–316.
- [57] A. A. Shkalikov, On the basis problem of the eigenfunctions of an ordinary differential operator, *Russ. Math. Surv.* **34** (5) (1979), 249–250.
- [58] A. A. Shkalikov, The basis problem of the eigenfunctions of ordinary differential operators with integral boundary conditions. *Moscow Univ. Math. Bull.* (6) **37** (1982), 10–20.
- [59] A. A. Shkalikov, Boundary Problems for Ordinary Differential Equations with Parameter in the Boundary Conditions, *Tr. Semin. im. I.G. Petrovskogo* **9** (1983), 190–229; English transl. in *J. Soviet Math.* (6) **33** (1986), 1311–1342.
- [60] A. A. Shkalikov, On the basis property of root vectors of a perturbed self-adjoint operator. *Proc. Steklov Inst. Math.* **269** (2010), 284–298.
- [61] A. A. Shkalikov and O. A. Veliev, On the Riesz Basis Property of the Eigen- and Associated Functions of Periodic and Antiperiodic Sturm-Liouville Problems. *Mathematical Notes* (5) **85**, (2009), 647–660.
- [62] M. A. Shubov, Asymptotic and spectral analysis of the spatially nonhomogeneous Timoshenko beam model. *Math. Nachr.* **241** (2002), 125–162.
- [63] A. Soufyane and A. Wehbe, Uniform stabilization for the Timoshenko beam by a locally distributed damping, *Electronic J. Differ. Equ.*, **2003** (29) (2003), 1–14.
- [64] J.D. Tamarkin, Sur quelques points de la theorie des equations differentielles lineaires ordinaires et sur la generalisation de la serie de Fourier, *Rend. Circ. Mat. Palermo* **34** (2) (1912), 345–382.
- [65] J. D. Tamarkin, *On some general problems of the theory of ordinary linear differential operators and the expansion of arbitrary functions into series*. Petrograd 1917.
- [66] J. D. Tamarkin, Some general problems of the theory of linear differential equations and expansions of an arbitrary functions in series of fundamental functions. *Math. Z.* **27** (1928), 1–54.
- [67] S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical magazine* **41** (1921), 744–746.
- [68] S. Timoshenko, *Vibration Problems in Engineering*. Van Norstrand, New York 1955.
- [69] I. Trooshin and M. Yamamoto, Riesz basis of root vectors of a nonsymmetric system of first-order ordinary differential operators and application to inverse eigenvalue problems, *Appl. Anal.* **80** (2001), 19–51.
- [70] I. Trooshin and M. Yamamoto, Spectral properties and an inverse eigenvalue problem for nonsymmetric systems of ordinary differential operators. *J. Inverse Ill-Posed Probl.* (6) **10** (2002), 643–658.
- [71] J. Wermer, On invariant subspaces of normal operators, *Proc. Amer. Math. Soc.* **3** (2) (1952), pp. 270–277.
- [72] Y. Wu and X. Xue, Decay rate estimates for the quasi-linear Timoshenko system with nonlinear control and damping terms. *J. Math. Physics* 093502 **52** (2011), 18 p.
- [73] G. Q. Xu, Z. J. Han and S. P. Yung, Riesz basis property of serially connected Timoshenko beams. *Inter. J. Control* (3) **80** (2007), 470–485.
- [74] G. Q. Xu and S. P. Yung, Exponential Decay Rate for a Timoshenko Beam with Boundary Damping. *J. Optimiz. Theory Appl.* (3) **123** (2004), 669–693.

INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, NAS OF UKRAINE, R. LUXEMBURG STR. 74, 83114  
DONETSK, UKRAINE

*E-mail address:* A.A.Lunyov@gmail.com

INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, NAS OF UKRAINE, R. LUXEMBURG STR. 74, 83114  
DONETSK, UKRAINE

*E-mail address:* mmm@telenet.dn.ua